

Symmetries, conservation laws, and cohomology of Maxwell's equations using potentials

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Abstract

New nonlocal symmetries and conservation laws are derived for Maxwell's equations in 3+1 dimensional Minkowski space using a covariant system of joint vector potentials for the electromagnetic tensor field and its dual. A key property of this system, as well as of this class of new symmetries and conservation laws, is their invariance under the duality transformation that exchanges the electromagnetic field with its dual. (In contrast the standard potential system using a single vector potential is not duality-invariant.) The nonlocal symmetries of Maxwell's equations come from an explicit classification of all symmetries of a certain natural geometric form admitted by the joint potential system in Lorentz gauge. In addition to scaling and duality-rotation symmetries, and the well-known Poincaré and dilation symmetries which involve homothetic Killing vectors, the classification yields new geometric symmetries involving Killing-Yano tensors related to rotations/boosts and inversions. The nonlocal conservation laws of Maxwell's equations are constructed from these geometric symmetries by applying a conserved current formula that uses the joint potentials and directly generates conservation laws from any (local or nonlocal) symmetries of Maxwell's equations. This formula is shown to arise through a series of mappings that relate, respectively, symmetries/adjoint-symmetries of the joint potential system and adjoint-symmetries/symmetries of Maxwell's equations. The mappings are derived as by-products of the study of cohomology of closed 1-forms and 2-forms locally constructed from the electromagnetic field and its derivatives to any finite order for all solutions of Maxwell's equations. In particular it is shown that the only nontrivial cohomology consists of the electromagnetic field (2-form) itself as well as its dual (2-form), and that this 2-form cohomology is killed by the introduction of corresponding potentials.

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I. INTRODUCTION

Two basic aspects in the study of field equations in mathematical physics are their symmetry structure and their conservation law structure. Geometrically speaking, symmetries are infinitesimal transformations of the fields under which all solutions are mapped into solutions. Symmetries of local form in the fields and partial derivatives of the fields to a finite order (generalizing classical point symmetries) are the basis for construction of exact (invariant) solutions and provide an important connection with separation of variables in certain cases. In the situation where a system of field equations has a Lagrangian, every local symmetry that leaves the Lagrangian invariant to within a divergence yields a conservation law, namely a current (density and flux) whose space-time divergence (i.e. time derivative of density plus spatial divergence of flux) vanishes on all solutions of the field equations, through Noether's theorem. Conservation laws given by currents of local form determine physically important conserved quantities such as energy, momentum, angular momentum, mass, charge, etc. which are constants of motion central to an analysis of the time evolution of the fields. An infinite hierarchy of local symmetries and conservation laws of increasing higher order is a hallmark of complete integrability of field equations. Nonlocal symmetries and nonlocal conservation laws, involving other than a local form, such as essential dependence on potentials or integrals of the fields, have been less studied but are also important and useful in the study of field equations — for instance, they yield exact solutions and conserved quantities that are not obtainable from local symmetries or local conservation laws [1].

For the source-free Maxwell's equations in classical electromagnetic field theory in 3+1 dimensional Minkowski space, a complete explicit classification of all local symmetries and local conservation laws in a unified coordinate-invariant form has been recently carried out by Anco & Pohjanpelto [2, 3]. The symmetry classification was obtained by solving the symmetry determining equations through the use of spinor techniques (and properties of Killing spinors). Since Maxwell's equations are a non-Lagrangian system, Noether's theorem cannot be used to find conservation laws from symmetries. A scaling formula that generates all non-trivial conserved currents from adjoint-symmetries was instead used to classify conservation laws through solving the similar adjoint-symmetry determining equations. This formula is a variant of a general conservation law formula that applies to any PDE system admitting a

scaling symmetry and produces all local conserved currents with nonzero scaling weight [4].

The local conservation laws of Maxwell's equations comprise the well-known stress-energy currents and Lipkin's zilch currents [5, 6, 7, 8, 9], and new chiral currents which contain first derivatives of the electromagnetic field and hence are of one order higher in derivatives compared to the stress-energy currents. Associated with these conservation laws are corresponding conserved tensors. The local symmetries of Maxwell's equations consist of infinitesimal scaling and duality-rotation transformations, infinitesimal Poincaré, dilation and conformal transformations [10, 11, 12], and in addition infinitesimal chiral transformations which are, again, of one order higher in derivatives. Second order symmetries and conserved currents of chiral type were first discovered by Fushchich & Nikitin [13, 14, 15] several years ago and possess the striking feature of odd parity under exchange of electric and magnetic fields, in contrast to the even parity of the stress-energy and zilch currents as well as that of the Poincaré symmetries and the dilation/conformal symmetries. Due to the covariant linear nature of Maxwell's equations, a hierarchy of higher order symmetries and currents arise [16] by repeated replacement of the electromagnetic field by Poincaré and dilation/conformal symmetry operators applied to the field or its dual. The general classification results obtained in Ref.[2, 3] state that no other local symmetries or local conservation laws exist to all orders, apart from elementary ones of zeroth order (which are produced through shifting the electromagnetic field by any particular solution of Maxwell's equations).

Some nonlocal symmetries and conservation laws for Maxwell's equations were also derived by Fushchich & Nikitin [13, 14], in a non-covariant manner using Fourier transform methods. For the reduction of Maxwell's equations to 2+1 dimensions, nonlocal symmetries and conservation laws in covariant form were obtained by Anco & Bluman [17] through the use of electric and magnetic potentials. The well-known covariant vector potential for Maxwell's equations in any number of dimensions gives a Lagrangian system with gauge freedom. Note that this potential arises from the absence of magnetic charges and currents in free space and hence we refer to it as the magnetic vector potential. In Lorentz gauge the magnetic potential system reduces to the vector wave equation. If electric charges and currents are absent, as in the source-free Maxwell's equations, there is an electric potential system analogous to the magnetic one. In 3+1 dimensions, the electric and magnetic potentials are each covariant vector fields that are dual in the sense that they are exchanged under a duality transformation on the electromagnetic field. By comparison, in 2+1 dimensions

the electric potential is a scalar field satisfying the ordinary wave equation and duality is lost. (In more than three space dimensions, duality is also lost since the electric potential becomes an antisymmetric tensor field.)

The introduction of these potentials for Maxwell's equations is an instance of Bluman's method of potential systems [18, 19, 20] and Vinogradov's theory of coverings for PDE systems [21, 22, 23]. In these two approaches, nonlocal symmetries of a given PDE system are realized as local symmetries of a potential system (or covering system). Potential systems are characterized by the embedding property [19] that modulo gauge freedom their solutions are in one-to-one correspondence with solutions of the given PDE system. (Gauge freedom, meaning a local symmetry that depends on an arbitrary function of all independent variables, arises automatically for potentials only in more than one space dimension.) It is essential to have a gauge imposed on potentials in order to obtain nonlocal symmetries, because as proved by Anco & Bluman [17] when a potential system possesses gauge freedom then all of its local symmetries project onto only local (gauge-invariant) symmetries of the original PDE system (provided the system is locally well-posed in the sense that it is locally solvable [24] but has no solutions involving an arbitrary function of all independent variables). Furthermore, an extension of their proof shows that all local conservation laws of such a potential system project onto conserved currents whose form is necessarily gauge invariant modulo terms that are trivially divergence-free, which presents a severe limitation for obtaining nonlocal conservation laws with essential dependence on potentials.

In this paper we use a natural joint potential system with Lorentz gauge imposed to obtain new nonlocal symmetries and nonlocal conservation laws of Maxwell's equations in 3+1 dimensions. This potential system involves the simultaneous introduction of both electric and magnetic vector potentials. Thus, it inherits the electric-magnetic duality invariance of Maxwell's equations and similarly is non-Lagrangian. To our knowledge, there has been no previous systematic investigation of the symmetries or conservation laws of Maxwell's equations using these joint potentials.

After setting out some preliminaries in Sec. II, we discuss some important interrelationships among symmetries, conservation laws, and adjoint-symmetries of Maxwell's equations and its various potential systems without gauges in Sec. III. These interrelationships come from the local p -form cohomology of Maxwell's equations, i.e. p -forms locally constructed from the spacetime coordinates and the electromagnetic field and its derivatives

(to some finite order) on all solutions. We show that the cohomology of closed p -forms modulo exact p -forms determines mappings between symmetries and adjoint-symmetries. Since adjoint-symmetries generate conserved currents through a scaling formula [3, 4], we obtain a correspondence between conservation laws and both symmetries and adjoint-symmetries. Most importantly, this leads to an explicit formula that generates conserved currents directly from any symmetries of Maxwell's equations or its joint potential system (thus by-passing the absence of a Lagrangian). We apply these results to the well-known geometric symmetries of Maxwell's equations and their counterparts for the potential systems without gauges. Our results explicitly demonstrate how these potential systems yield only local symmetries and local conserved currents of Maxwell's equations, with the exception of one conserved current generated from the duality-rotation symmetry of the joint potential system. This current has an essential dependence on the joint potentials yet is found to be invariant with respect to the gauge freedom in these potentials modulo trivially conserved terms.

In Sec. IV we investigate the joint potential system with Lorentz gauge imposed. A classification of geometric symmetries is derived by solving the symmetry determining equations using covariant tensorial methods, from which we obtain new local symmetries along with corresponding new local conservation laws in terms of the potentials. As main results, in Sec. V we show that these symmetries and conservation laws are nonlocal under projection to Maxwell's equations, and we discuss some of their resulting features. We make some concluding remarks in Sec. VI.

II. PRELIMINARIES

Maxwell's equations for the electromagnetic field tensor $F_{\mu\nu}(x) = F_{[\mu\nu]}(x)$ in Minkowski space $M^4 = (\mathbb{R}^4, \eta)$ are given by

$$\partial_\mu F^{\mu\nu}(x) = 4\pi J_e^\nu(x), \quad \partial_\mu *F^{\mu\nu}(x) = 4\pi J_m^\nu(x), \quad (2.1)$$

with electric and magnetic current sources. Here

$$*F_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\sigma\tau}F^{\sigma\tau} \quad (2.2)$$

is the dual of $F_{\mu\nu}$, $\epsilon_{\mu\nu\sigma\tau}$ is the spacetime volume form, x^μ are the standard Minkowski coordinates, and $\partial_\mu = \partial/\partial x^\mu$ is the coordinate derivative. Throughout, indices will be

freely lowered or raised using the spacetime metric $\eta_{\mu\nu}$ and its inverse $\eta^{\mu\nu}$ (with signature $(-+++)$). When there are no current sources, the field equations

$$\partial^\mu F_{\mu\nu}(x) = 0, \quad \partial^\mu *F_{\mu\nu}(x) = 0 \quad (2.3)$$

display invariance under the duality transformation

$$F_{\mu\nu} \rightarrow *F_{\mu\nu}, \quad *F_{\mu\nu} \rightarrow -F_{\mu\nu}. \quad (2.4)$$

To introduce potential variables in a covariantly natural way, we rewrite the source-free Maxwell's equations (2.3) in the equivalent form

$$\partial_{[\sigma} F_{\mu\nu]}(x) = 0, \quad \partial_{[\sigma} *F_{\mu\nu]}(x) = 0. \quad (2.5)$$

Since $F(x) = F_{\mu\nu}(x)dx^\mu \wedge dx^\nu$ is a closed 2-form and Minkowski space is topologically trivial, we can conclude by Poincaré's Lemma that $F(x)$ is exact, i.e.

$$F_{\mu\nu}(x) = \partial_{[\mu} A_{\nu]}(x) \quad (2.6)$$

for some 1-form potential $A(x) = A_\nu(x)dx^\nu$. The standard *magnetic potential system* is given by

$$\partial^\mu \partial_{[\mu} A_{\nu]}(x) = 0 \quad (2.7)$$

which is a self-adjoint system and thus arises from a Lagrangian. This system (2.7) possesses gauge freedom

$$A_\nu(x) \rightarrow A_\nu(x) + \partial_\nu \chi(x), \quad (2.8)$$

where $\chi(x)$ is an arbitrary function of x^μ .

The field equations (2.5) also imply that $*F(x) = *F_{\mu\nu}(x)dx^\mu \wedge dx^\nu$ is a closed 2-form. So again by Poincaré's Lemma, $*F(x)$ is exact, i.e.

$$*F_{\mu\nu}(x) = \partial_{[\mu} A'_{\nu]}(x), \quad (2.9)$$

for some 1-form potential $A'(x) = A'_\nu(x)dx^\nu$ which satisfies an *electric potential system* analogous to the magnetic potential system for $A(x)$. Since by duality the electric potential system shares all the same properties as the magnetic potential system, we will omit it in our subsequent discussion and results.

A further natural potential system of Maxwell's equations (2.3) is obtained by introducing both electric and magnetic potentials simultaneously. Since $F_{\mu\nu}(x) = \partial_{[\mu} A_{\nu]}(x)$ and

$*F_{\mu\nu}(x) = \partial_{[\mu}A'_{\nu]}(x)$ must satisfy the duality relation (2.2), we define their joint *electric-magnetic potential system* to be

$$\partial_{[\mu}A'_{\nu]}(x) = \frac{1}{2}\epsilon_{\mu\nu\sigma\tau}\partial^{\sigma}A^{\tau}(x). \quad (2.10)$$

The duality transformation (2.4) on the electromagnetic field induces a corresponding duality transformation

$$A_{\mu} \rightarrow A'_{\mu}, \quad A'_{\mu} \rightarrow -A_{\mu} \quad (2.11)$$

on the potentials. (Note that putting $A' = \mathbf{1}A$ would give the self-dual Maxwell's equations.)

The electric-magnetic potential system (2.10) admits the gauge freedom

$$A_{\nu}(x) \rightarrow A_{\nu}(x) + \partial_{\nu}\chi(x), \quad A'_{\nu}(x) \rightarrow A'_{\nu}(x) + \partial_{\nu}\chi'(x), \quad (2.12)$$

where $\chi(x)$ and $\chi'(x)$ are arbitrary functions of x^{μ} . Unlike the standard potential system (2.7), the joint system (2.10) is not self-adjoint, and hence it is a non-Lagrangian system.

It is important to note that via the embedding relations (2.6), (2.9), and the duality relation (2.2), the solutions of the potential systems (2.7) and (2.10) modulo gauge freedom are in one-to-one correspondence with the solutions of Maxwell's equations (2.3).

Associated with Maxwell's equations or any of its potential systems is the respective jet space J^q of order $0 \leq q \leq \infty$ defined as the coordinate manifold such that each point (q -jet) in J^q is identified with a spacetime point x and the values of the field or potential(s) and its partial derivatives up to order q at x . Note here that the jet space J^0 is identified with $M^4 \times E$ where E is the vector space of 2-forms for the case of Maxwell's equations, 1-forms for the case of the magnetic potential system, and pairs of 1-forms in the case of the joint potential system. In this setting we use D_{μ} to denote the total derivative operator with respect to x^{μ} and write a subscript “, μ ” for coordinates corresponding to differentiation by D_{μ} on the field or potential(s) in the standard way [3, 16]. For Maxwell's equations the space of solutions is represented by the submanifold (solution jet space) $R(F) \subset J^1(F)$ whose coordinates quotient out the field equations on the first-order partial derivatives of the electromagnetic field in $J^1(F)$ [3, 16]. The q -prolonged solution jet space $R^q(F) \subset J^{q+1}(F)$ of Maxwell's equations is defined by a similar quotient with respect to the q th-order partial derivatives of the electromagnetic field equations, $1 \leq q \leq \infty$. There is an analogous construction of (prolonged) solution jet spaces $R^q(A, A') \subset J^{q+1}(A, A')$ in the case of the joint potential system, and $R^q(A) \subset J^{q+2}(A)$ in the case of the magnetic potential system

[16]. Note explicit coordinates for $R^0(F) := R(F)$ consist of $(x^\mu, F_{\mu\nu}, \text{trfr} F_{\mu(\nu, \sigma)})$; likewise $(x^\mu, A_\nu, A'_\nu, F_{\mu\nu}, A_{(\nu, \mu)}, A'_{(\nu, \mu)})$ and $(x^\mu, A_\nu, F_{\mu\nu}, A_{(\nu, \mu)}, A_{(\nu, \mu\sigma)}, \text{trfr} F_{\mu(\nu, \sigma)})$ are coordinates for $R^0(A, A') := R(A, A')$ and $R^0(A) := R(A)$, where “trfr” on a tensor stands for its totally trace-free part with respect to the Minkowski metric. Similar coordinates can be written down for the prolonged solution spaces to all orders, representing those components of the electromagnetic field, potentials, and their partial derivatives that are freely specifiable at a spacetime point. Throughout, we indicate jet space coordinates by writing (derivatives of) F, A, A' without (x) dependence.

It would be typical to proceed by defining symmetries as infinitesimal transformations on the (prolonged) solution jet space R^∞ . For our purposes, an equivalent characterization of symmetries via determining equations [25] is better suited.

For Maxwell’s equations or its potential systems, local symmetries of order $q < \infty$ are characterized by E -valued functions on $J^q \subset J^\infty$ (in the given coordinates) whose restriction to R^∞ satisfies the linearization of the system equations. Similarly, local adjoint-symmetries of order q are characterized by \tilde{E} -valued functions on $J^q \subset J^\infty$ whose restriction to R^∞ satisfies the adjoint linearization of the system equations, where \tilde{E} is the vector space of 2-forms in the case of the joint potential system, 1-forms in the case of the magnetic potential system, and pairs of 1-forms in the case of Maxwell’s equations. A symmetry or adjoint-symmetry of order q is trivial if it vanishes when evaluated on R^∞ ; two symmetries or adjoint-symmetries that differ by a trivial one are considered to be equivalent.

System	Symmetry Equations	Adjoint-symmetry Equations
Maxwell’s equations $D^\mu F_{\mu\nu} = 0$ $\frac{1}{2}\epsilon_{\mu\nu\sigma\tau} D^\mu F^{\sigma\tau} = 0$	$D^\mu P_{\mu\nu} = 0$ $\frac{1}{2}\epsilon_{\mu\nu\sigma\tau} D^\mu P^{\sigma\tau} = 0$	$D_{[\mu} Q'_{\nu]} = \frac{1}{2}\epsilon_{\mu\nu\sigma\tau} D^\sigma Q^\tau$
Magnetic potential system $D^\mu D_{[\mu} A_{\nu]} = 0$	$D^\mu D_{[\mu} Q_{\nu]} = 0$	$D^\mu D_{[\mu} Q'_{\nu]} = 0$
Joint potential system $D_{[\mu} A'_{\nu]} = \frac{1}{2}\epsilon_{\mu\nu\sigma\tau} D^\sigma A^\tau$	$D_{[\mu} Q'_{\nu]} = \frac{1}{2}\epsilon_{\mu\nu\sigma\tau} D^\sigma Q^\tau$	$D^\mu P_{\mu\nu} = 0$ $\frac{1}{2}\epsilon_{\mu\nu\sigma\tau} D^\mu P^{\sigma\tau} = 0$

TABLE I: Symmetry and adjoint-symmetry equations for Maxwell’s equations and potential systems

Viewed geometrically [26], a local symmetry of Maxwell's equations or its potential systems describes a generalized vector field \mathbf{X} on $M^4 \times E$ whose prolongation $\text{pr}\mathbf{X}$ to J^∞ is tangent to the solution jet space R^∞ and involves no motion on the spacetime coordinates x^μ . In particular, corresponding to $P_{\mu\nu}, Q_\mu, (Q_\mu, Q'_\mu)$ appearing in the symmetry equations in Table I we have the generators

$$\mathbf{X} = P_{\mu\nu} \frac{\partial}{\partial F_{\mu\nu}}, \quad \mathbf{X} = Q_\mu \frac{\partial}{\partial A_\mu}, \quad \mathbf{X} = Q_\mu \frac{\partial}{\partial A_\mu} + Q'_\mu \frac{\partial}{\partial A'_\mu}. \quad (2.13)$$

This definition can be obviously generalized to allow motion on x^μ but it is well-known that every such symmetry is equivalent to one (referred to as its evolutionary form) without any motion on x^μ [24, 26]. In contrast, unlike for symmetries, there is no obvious geometrical meaning for adjoint-symmetries unless the system equations are self-adjoint (in which case adjoint-symmetries coincide with symmetries).

A local conserved current of order $q < \infty$ for Maxwell's equations or its potential systems is a vector function Ψ^μ on $J^q \subset J^\infty$ whose divergence vanishes on the solution jet space, namely

$$D_\mu \Psi^\mu = 0 \quad \text{on } R^\infty. \quad (2.14)$$

A current is trivial if it is equal to a curl $\Psi^\mu = D_\nu \Theta^{\mu\nu}$ when evaluated on R^∞ , where $\Theta^{\mu\nu} = \Theta^{[\mu\nu]}$ is some skew-tensor function on J^r , $q \leq r < \infty$. Two currents that differ by a trivial one are considered to be equivalent. The equivalence class of conserved currents containing a current Ψ^μ is called the conservation law associated with Ψ^μ . A conservation law has order q if the minimum order among all conserved currents in its equivalence class is equal to q .

For Maxwell's equations and its potential systems, the explicit coordinates introduced earlier for R and its prolongations can be used to show that all nontrivial currents of order q modulo curls are characterized by multipliers (also called characteristics) such that the divergence of a current $D_\mu \Psi^\mu$ on J^{q+1} yields, after integration by parts where necessary, a contracted product of the multipliers and the system equations. It is well-known from general results (see for instance [27]) that multipliers of order q are adjoint-symmetries subject to certain conditions on their adjoint-linearization on J^q , and there is a homotopy integral formula to recover a current (modulo a curl) from its multipliers. Two main complications arise in dealing with the correspondence between multipliers and currents for Maxwell's

equations. Firstly, because it is not a PDE system of Cauchy-Kovalevskaya form, the simple relation [24, 28] that two currents are equivalent if and only if their multipliers agree on R^∞ breaks down and there exist trivial currents $\Psi^\mu = D_\nu(F^{\mu\nu}\chi + *F^{\mu\nu}\chi')$ of order $q + 1$ whose multipliers are a class of nontrivial adjoint-symmetries $Q_\nu = D_\nu\chi$, $Q'_\nu = D_\nu\chi'$, which do not vanish on R^∞ , for any (non-constant) functions χ, χ' on J^q . Secondly, due to the linear nature of Maxwell's equations, at every order $q > 0$ there are other classes of nontrivial adjoint-symmetries all of which fail to satisfy the adjoint-linearization conditions even to within the addition of a trivial adjoint-symmetry. Similar complications are found to occur for the magnetic and joint potential systems.

However, an alternative way of generating all nontrivial conserved currents by-passing these complications for Maxwell's equations and its potential systems is provided by a general scaling formula that produces conserved currents directly from adjoint-symmetries [3, 4]. This formula is derived from the adjoint relation between the determining equations for symmetries and adjoint-symmetries of any given PDE system [27]. The resulting conserved current formulas for Maxwell's equations and its potential systems are displayed in Table II; the notation $|_{\lambda F}$ is used to denote a scaling of F and derivatives of F in functions on $J^q(F)$ by a parameter λ , and likewise for functions on $J^q(A)$ or $J^q(A, A')$.

System	Adjoint-symmetry	Conserved current formula
Maxwell's equations $D^\mu F_{\mu\nu} = 0$ $D^\mu *F_{\mu\nu} = 0$	Q_ν, Q'_ν on $R^q(F)$	$\Psi^\mu = \int_0^1 \left(Q_\nu F^{\mu\nu} + Q'_\nu *F^{\mu\nu} \right) \Big _{\lambda F} \frac{d\lambda}{\lambda}$
Magnetic potential system $D^\mu D_{[\mu} A_{\nu]} = 0$	Q_ν on $R^q(A)$	$\Psi^\mu = \int_0^1 \left(Q_\nu D^{[\mu} A^{\nu]} - D^{[\mu} Q^{\nu]} A_\nu \right) \Big _{\lambda A} \frac{d\lambda}{\lambda}$
Joint potential system $D_{[\mu} A'_{\nu]} = *D_{[\mu} A_{\nu]}$	$P_{\mu\nu}$ on $R^q(A, A')$	$\Psi^\mu = \int_0^1 \left(P^{\mu\nu} A'_\nu - *P^{\mu\nu} A_\nu \right) \Big _{\lambda A, \lambda A'} \frac{d\lambda}{\lambda}$

TABLE II: Conserved current formulas for Maxwell's equations and potential systems

General results in Ref.[4] establish a key property of these conserved current formulas. A direct proof in the case of Maxwell's equations was given in Ref.[3].

Proposition II.1 *For an adjoint-symmetry that agrees with a multiplier on the solution jet space, the conserved current scaling formula generates an equivalent nontrivial current;*

otherwise for an adjoint-symmetry differing from any multiplier on the solution jet space, it yields a trivial conserved current.

In the case of the magnetic potential system, which is self-adjoint and thus arises from a Lagrangian, adjoint-symmetries are the same as symmetries, and multipliers correspond to those symmetries under which the Lagrangian of the system is invariant (to within a divergence). The scaling formula in the Lagrangian case produces a current equivalent to the one that comes from Noether's theorem applied to such symmetries.

III. COHOMOLOGY, LOCAL SYMMETRIES AND LOCAL CONSERVATION LAWS

In this section we present a unified account of interrelationships (i.e. mappings) among local symmetries, local adjoint-symmetries, and local conservation laws of Maxwell's equations, and the magnetic and joint potential systems on Minkowski space. The cohomology of differential 1-forms and 2-forms on the solution jet space of these systems together with a locality-projection theorem for local symmetries will be the main tools in the derivation of these results.

We will consistently use P to denote a differential 2-form, Q a differential 1-form, χ a scalar function (0-form), e.g. on $J^q(F)$,

$$P = P_{\mu\nu}[F]dx^\mu \wedge dx^\nu, \quad Q = Q_\mu[F]dx^\mu, \quad \chi = \chi[F], \quad (3.1)$$

where dependence on jet space coordinates to some finite order is denoted by $[F]$. A similar notation will be used for differential forms on $J^q(A)$ and $J^q(A, A')$ and when these variables carry primes, tildes, etc. The total differential will be denoted by D , e.g.

$$DP = D_{[\sigma}P_{\mu\nu]}[F]dx^\sigma \wedge dx^\mu \wedge dx^\nu, \quad DQ = D_{[\mu}Q_{\nu]}[F]dx^\mu \wedge dx^\nu, \quad D\chi = D_\mu\chi[F]dx^\mu. \quad (3.2)$$

In differential form notation, the system equations and determining equations for symmetries and adjoint-symmetries are summarized in Table III.

Associated with the magnetic and joint potential systems, there is a natural embedding of the respective jet spaces $J^{q+1}(A)$ and $J^{q+1}(A, A')$ into the jet space $J^q(F)$ under the total differential mapping given by $F = DA = -*DA'$ and its obvious prolongation ($0 \leq q \leq \infty$).

System	Symmetry Equations	Adjoint-symmetry Equations
$DF = 0$ $D*F = 0$	$DP[F] = 0$ $D*P[F] = 0$	$DQ'[F] = *DQ[F]$
$D*DA = 0$	$D*DQ[A] = 0$	$D*DQ'[A] = 0$
$DA' = *DA$	$DQ'[A, A'] = *DQ[A, A']$	$DP[A, A'] = 0$ $D*P[A, A'] = 0$

TABLE III: Symmetry and adjoint-symmetry equations for Maxwell's equations and potential systems

This embedding extends to the prolonged solution jet spaces $R^\infty \subset J^\infty$ and is linear and one-to-one modulo the gauge freedom on the potentials, namely $F = 0$ if and only if $A = D\chi$, $A' = D\chi'$, where χ, χ' are any scalar functions on J^q .

We note the following consequences of gauge freedom in the potential systems here. The (adjoint-) symmetry determining equations of the standard potential system have solutions of the form $Q[A] = D\chi[A]$ (i.e. gauge symmetries on A) for an *arbitrary* scalar function χ on $R^q(A)$. Such solutions can be also viewed as representing a gauge freedom in the form of $Q[A]$. Similarly, the symmetry determining equations of the joint potential system have solutions $Q[A, A'] = D\chi[A, A']$, $Q'[A, A'] = D\chi'[A, A']$ (i.e. gauge symmetries on A, A') for a pair of *arbitrary* scalar functions χ, χ' on $R^q(A, A')$; the adjoint-symmetry determining equations of this system can be shown to have no gauge freedom in the form of $P[A, A']$ (basically, the latter equations are locally well-posed as a PDE system for P). By comparison with the corresponding determining equations for symmetries and adjoint-symmetries of Maxwell's equations, it follows that $Q[F] = D\chi[F]$, $Q'[F] = D\chi'[F]$ represent gauge freedom in the form of adjoint-symmetries, for pairs of *arbitrary* scalar functions χ, χ' on $R^q(F)$, while there is no gauge freedom in the form of the symmetries (since Maxwell's equations are a locally well-posed PDE system).

The total differential D obviously satisfies $D^2 = 0$ and hence defines a complex of differential forms on J^∞ , which is a generalization of the de Rham complex on Minkowski space to the jet space setting [29]. What we will refer to as differential forms on R^q can be defined more precisely as the pullback to R^∞ (via the inclusion map) of differential forms on J^∞ whose coefficients depend on the jet coordinates up to a finite order $q + 1$ in the case of

Maxwell's equations and the joint potential system; $q + 2$ in the case of the magnetic potential system. For notational convenience, we will also denote the induced total differential mapping on R^∞ by D and hence obtain the D -complex on R^∞ . Results concerning the local cohomology of this complex, i.e. D -closed modulo D -exact differential forms, will underlie our study of the local symmetries and adjoint-symmetries of Maxwell's equations and its potential systems. It is worth remarking that, in contrast, off R^∞ the local cohomology of the (free) D -complex is trivial, since (see [29]) all D -closed differential p -forms on J^∞ for $1 \leq p < 4$ are D -exact.

A. Cohomology and locality projection

We now state the cohomology and locality-projection theorems and then give their proofs afterwards.

Theorem III.1 (Local Cohomology)

1-form cohomology: Let Q be a differential 1-form on R^q , $0 \leq q < \infty$. If Q is closed, i.e. $DQ = 0$, then

$$(i) \quad DF = D*F = 0 \implies Q[F] = D\chi[F]; \quad (3.3)$$

$$(ii) \quad D*DA = 0 \implies Q[A] = D\chi[A]; \quad (3.4)$$

$$(iii) \quad DA' = *DA \implies Q[A, A'] = D\chi[A, A']. \quad (3.5)$$

Thus, the local 1-form cohomology is trivial for each of these systems.

2-form cohomology: Let P be a differential 2-form on R^q , $0 \leq q < \infty$. If P is closed, i.e. $DP = 0$, then

$$(i) \quad DF = D*F = 0 \implies P[F] = c_1 F + c_2 *F + DQ[F]; \quad (3.6)$$

$$(ii) \quad D*DA = 0 \implies P[A] = c *F + DQ[A], \quad F = DA; \quad (3.7)$$

$$(iii) \quad DA' = *DA \implies P[A, A'] = DQ[A, A']; \quad (3.8)$$

*for some constants c, c_1, c_2 . Thus, F and $*F$ represent the only nontrivial local 2-form cohomology, and this cohomology is killed by the introduction of a corresponding potential.*

We remark that local 1-form and 2-form cohomology has a field-theoretic interpretation in terms of conserved charges [30], describing electromagnetic fluxes through closed loops and

surfaces in spacetime. (Namely, the integral of $*F(x), F(x)$ over any closed surface yields the total magnetic and electric charge enclosed within the surface; in Minkowski space, these charges vanish for all smooth solutions $F(x)$ of Maxwell's equations.)

Theorem III.2 (Locality Projection)

*Let Q be a differential 1-form on R^q , $0 \leq q < \infty$. If the 2-form $*DQ$ is closed, i.e. $D*DQ = 0$, then*

$$(i) \quad D*DA = 0 \implies DQ[A] = P[F], \text{ where } F = DA; \quad (3.9)$$

$$(ii) \quad DA' = *DA \implies DQ[A, A'] = P[F], \text{ where } F = DA = -*DA'. \quad (3.10)$$

Thus, any essential dependence on potentials in Q is killed under total exterior differentiation.

A proof of the local 2-form cohomology theorem for Maxwell's equations in the case where P is a linear function on $R^q(F)$ is given in Ref.[31] using tensorial methods. This proof amounts to showing that the vanishing of the cohomology equation (3.6) has no non-trivial linear solutions. The nonlinear case can be reduced to the linear case by standard linearization techniques and the whole computation is especially tractable in spinor form with the methods used in Ref.[3]. The proof for the potential systems and the local 1-form cohomology theorem can be done by the same techniques.

The locality-projection theorem is a consequence of applying a general result proved by Anco & Bluman [17]: for a locally well-posed PDE system (i.e. if it is locally solvable [24] such that no solutions depend on an arbitrary function of all independent variables), the local symmetries of any potential system with gauge freedom project onto only local symmetries under the embedding of the solution space of the potential system into the solution space of the given PDE system. In particular, no projected symmetries have any essential dependence on the potentials. If a differential 1-form Q on $R^q(A)$ satisfying $D*DQ = 0$ is viewed as a local symmetry of the magnetic potential system, then since Maxwell's equations is a locally well-posed system we immediately conclude that the projected symmetry $P = DQ$ for Maxwell's equations must be a differential 2-form on $R^{q-1}(F)$. A similar argument applies to a differential 1-form Q on $R^q(A, A')$. Because $D*DQ = 0$ implies that the differential 2-form $P' = *DQ$ is closed, the equation $*DQ = DQ'$ holds for some 1-form Q' on $R^q(A, A')$,

by the cohomology equation (3.8). If we then view the pair (Q, Q') as a local symmetry of the joint potential system, we again conclude $P = DQ$ must be a 2-form on $R^{q-1}(F)$.

It is crucial that both the cohomology and locality-projection theorems are formulated on the solution jet spaces R^q of finite order $q < \infty$. Indeed, on the infinite-order solution jet space R^∞ , these theorems break down in the following manner.

Proposition III.3 *On $R^\infty(F)$, the local 2-form cohomology becomes formally trivial:*

$$F = DA_F \quad \text{and} \quad *F = DA'_F \quad (3.11)$$

where

$$A_F = \sum_{k \geq 0} \frac{(-1)^k}{(k+1)!} x \lrcorner (\mathcal{D}^k F), \quad (3.12)$$

$$A'_F = \sum_{k \geq 0} \frac{(-1)^k}{(k+1)!} x \lrcorner (\mathcal{D}^k *F), \quad (3.13)$$

and $\mathcal{D} = x^\sigma D_\sigma$ denotes the dilation operator. Furthermore, these 1-forms satisfy Cronstrom's gauge

$$x \lrcorner A_F = x \lrcorner A'_F = 0. \quad (3.14)$$

The proof amounts to an explicit computation and will be omitted. We note that the formal series (3.12) and (3.13) arise from integration by parts of the Poincaré homotopy formula for the de Rham cohomology of differential forms on Minkowski space [24]. Thus, the notion of nonlocality or nontrivial cohomology associated with Maxwell's equations and its potential systems is meaningful only when we work in finite-order jet spaces.

B. Mappings and duality

We now give the statements of our main results which are consequences of the cohomology and locality-projection theorems. Throughout this section, it is understood that we work on finite-order solution jet spaces R^q , $0 \leq q < \infty$.

Theorem III.4 *The local symmetries and local adjoint-symmetries of Maxwell's equations and its potential systems have the following decompositions:*

$$(i) \quad DF = D*F = 0 \implies$$

$$P[F] = cF + c' *F + D\tilde{Q}[F], \quad *P[F] = c*F - c' F + D\tilde{Q}'[F], \quad (3.15)$$

$$Q[F] = \tilde{Q}[F] + D\chi[F], \quad Q'[F] = \tilde{Q}'[F] + D\chi'[F]; \quad (3.16)$$

$$(ii) \quad D*DA = 0 \implies$$

$$Q[A] = cA + \tilde{Q}[F] + D\chi[A], \quad (3.17)$$

$$Q'[A] = cA + \tilde{Q}'[F] + D\chi[A], \quad (3.18)$$

with $F = DA$;

$$(iii) \quad DA' = *DA \implies$$

$$Q[A, A'] = cA + c'A' + \tilde{Q}[F] + D\chi[A, A'], \quad Q'[A, A'] = cA' - c'A + \tilde{Q}'[F] + D\chi'[A, A'], \quad (3.19)$$

$$P[A, A'] = cF + c'*F + D\tilde{Q}[F], \quad *P[A, A'] = c*F - c'F + D\tilde{Q}'[F], \quad (3.20)$$

with $F = DA, *F = DA'$;

for some constants c, c' , where

$$D\tilde{Q}'[F] = *D\tilde{Q}[F] \quad (3.21)$$

in all cases. These decompositions are unique up to addition of arbitrary gradients $D\chi[F]$ to $\tilde{Q}[F]$, $D\chi'[F]$ to $\tilde{Q}'[F]$, and are stable under the duality invariance

$$P \rightarrow *P, \quad (Q, Q') \rightarrow (Q', -Q) \quad (3.22)$$

of the symmetry equations and adjoint-symmetry equations.

We mention that, moreover, the 1-forms $\tilde{Q}[F], \tilde{Q}'[F]$ here are canonically related under the duality transformation (2.4) on F , as will be discussed in Proposition III.13 later.

In the symmetry decompositions (3.15), (3.17), (3.19), the cohomology terms correspond to infinitesimal scaling and duality-rotation transformations

$$\mathbf{X}_{scal} = F \frac{\partial}{\partial F}, \quad A \frac{\partial}{\partial A}, \quad A \frac{\partial}{\partial A} + A' \frac{\partial}{\partial A'}, \quad (3.23)$$

$$\mathbf{X}_{dual} = *F \frac{\partial}{\partial F}, \quad A' \frac{\partial}{\partial A} - A \frac{\partial}{\partial A'}. \quad (3.24)$$

Note the duality-rotation is not realized as a local symmetry of the standard potential system. The symmetry-decomposition terms that involve χ or χ' correspond to a gauge freedom in the symmetries, namely the infinitesimal transformations

$$\mathbf{X}_{gauge} = D\chi \frac{\partial}{\partial A}, \quad D\chi \frac{\partial}{\partial A} + D\chi' \frac{\partial}{\partial A'} \quad (3.25)$$

for arbitrary functions χ, χ' on J^q are symmetries. The remaining terms in the symmetry decompositions represent gauge-invariant infinitesimal transformations

$$\mathbf{X} = D\tilde{Q}[F] \frac{\partial}{\partial F}, \quad \tilde{Q}[F] \frac{\partial}{\partial A}, \quad \tilde{Q}[F] \frac{\partial}{\partial A} + \tilde{Q}'[F] \frac{\partial}{\partial A'} \quad (3.26)$$

which are themselves symmetries. Note these terms are well-defined only up to gauge freedom (3.25) such that χ, χ' are functions on $J^q(F)$.

These results establish that the vector spaces of symmetries and adjoint-symmetries of each system are a direct sum of cohomology subspaces spanned by the separate terms proportional to c, c' , and a complementary subspace identified with the gauge-invariant non-cohomology terms involving \tilde{Q}, \tilde{Q}' , up to gradient terms $D\chi, D\chi'$. Hereafter we let X denote a vector space of symmetries, Y a vector space of adjoint-symmetries, and we use superscripts $c, c', 0$ to distinguish the respective subspaces in the direct sum decomposition; superscripts χ, χ' will denote the vector subspace defined by all $D\chi, D\chi'$ terms while a tilde will stand for the quotient with respect to this subspace. Thus we have the following vector space decompositions:

$$(i) \quad X_F = X_F^c \oplus X_F^{c'} \oplus X_F^0, \quad Y_F = Y_F^0, \quad (3.27)$$

$$(ii) \quad X_A = X_A^c \oplus X_A^0, \quad Y_A = Y_A^c \oplus Y_A^0, \quad (3.28)$$

$$(iii) \quad X_{A,A'} = X_{A,A'}^c \oplus X_{A,A'}^{c'} \oplus X_{A,A'}^0, \quad Y_{A,A'} = Y_{A,A'}^c \oplus Y_{A,A'}^{c'} \oplus Y_{A,A'}^0, \quad (3.29)$$

where the X^0, Y^0 subspaces in the case of differential 1-forms naturally partition into equivalence classes modulo gradients

$$(iv) \quad \tilde{Y}_F^0 = Y_F^0 / Y_F^{\chi, \chi'}, \quad \tilde{X}_A^0 = X_A^0 / X_A^\chi, \quad \tilde{Y}_A^0 = Y_A^0 / Y_A^\chi, \quad \tilde{X}_{A,A'}^0 = X_{A,A'}^0 / X_{A,A'}^{\chi, \chi'}. \quad (3.30)$$

It will be convenient to introduce a linear map $*'$ on differential 1-forms $\tilde{Q}[F] \mod D\chi[F]$ by the equation

$$D*'\tilde{Q}[F] = *D\tilde{Q}[F] \quad \text{on } R^q(F), \quad (3.31)$$

which defines an automorphism of each vector space (3.30). Note the relation (3.21) can be simply written $\tilde{Q}' = *'\tilde{Q}$.

Importantly, interrelationships hold between any two decompositions (3.15)–(3.20), given by linear maps summarized in the following four theorems.

Theorem III.5 (Self-correspondences related to the systems $DF = D*F = 0$, $DA' = *DA$, $D*DA = 0$)

There is a linear mapping between:

(i) *local symmetries on $R^q(F)$ and local adjoint-symmetries on $R^{q-1}(F)$, given by*

$$P[F] \mod F, *F \longleftrightarrow D(\tilde{Q}[F] \mod D\chi[F]), \quad (3.32)$$

$$*P[F] \mod F, *F \longleftrightarrow D(\tilde{Q}'[F] \mod D\chi'[F]), \quad (3.33)$$

corresponding to the isomorphism of vector spaces

$$X_F^0 \cong \tilde{Y}_F^0. \quad (3.34)$$

(ii) *local symmetries on $R^q(A, A')$ and local adjoint-symmetries on $R^{q+1}(A, A')$, given by*

$$D(Q[A, A'] \mod D\chi[A, A']) \longleftrightarrow P[A, A'], \quad (3.35)$$

$$D(Q'[A, A'] \mod D\chi'[A, A']) \longleftrightarrow *P[A, A'], \quad (3.36)$$

corresponding to the isomorphism of vector spaces

$$X_{A,A'}^c \cong Y_{A,A'}^c, \quad X_{A,A'}^{c'} \cong Y_{A,A'}^{c'}, \quad \tilde{X}_{A,A'}^0 \cong Y_{A,A'}^0. \quad (3.37)$$

(iii) *local symmetries on $R^q(A)$ and local adjoint-symmetries on $R^q(A)$, given by the direct identification*

$$Q[A] \longleftrightarrow Q'[A], \quad (3.38)$$

namely $X_A = Y_A$, as well as a dual identification

$$*'(Q[A] \mod A, D\chi[A]) \longleftrightarrow Q'[A] \mod A, D\chi[A], \quad (3.39)$$

corresponding to a nontrivial duality (isomorphism) of vector spaces $\tilde{X}_A^0 \cong \tilde{Y}_A^0$.

Note the composition of the maps (3.38) and (3.39) yields a linear mapping of local (adjoint-) symmetries modulo $A, D\chi[A]$ on $R^q(A)$ into themselves, corresponding to the vector space automorphism $*' : \tilde{X}_A^0 \longrightarrow \tilde{X}_A^0$ (and correspondingly $\tilde{Y}_A^0 \longrightarrow \tilde{Y}_A^0$).

Theorem III.6 (Correspondences related to the systems $D*DA = 0$ and $DA' = *DA$)

There is a linear mapping between:

(i) *local (adjoint-) symmetries on $R^q(A)$ and local symmetries on $R^q(A, A')$, given by*

$$Q[A] \text{ mod } D\chi[A] \longleftrightarrow Q[A, A'] \text{ mod } A', D\chi[A, A'], \quad (3.40)$$

$$*'Q[A] \text{ mod } D\chi[A] \longleftrightarrow Q'[A, A'] \text{ mod } A, D\chi'[A, A'], \quad (3.41)$$

where $'$ is extended by $*'A := A'$ and linearity so it is well-defined on all local (adjoint-) symmetries $Q[A]$ modulo gradients $D\chi[A]$, corresponding to the isomorphism of vector spaces*

$$X_A^c \cong X_{A,A'}^c, \quad \tilde{X}_A^0 \cong \tilde{X}_{A,A'}^0. \quad (3.42)$$

(ii) *local (adjoint-) symmetries on $R^q(A)$ and local adjoint-symmetries on $R^{q+1}(A, A')$, given by*

$$D(Q[A] \text{ mod } D\chi[A]) \longleftrightarrow P[A, A'] \text{ mod } *F, \quad (3.43)$$

corresponding to the isomorphism of vector spaces

$$X_A^c \cong Y_{A,A'}^c, \quad \tilde{X}_A^0 \cong Y_{A,A'}^0. \quad (3.44)$$

Theorem III.7 (Correspondences related to the systems $DF = D*F = 0$ and $DA' = *DA$)

There is a linear mapping between:

(i) *local symmetries on $R^q(F)$ and local adjoint-symmetries on $R^{q+1}(A, A')$, given by*

$$P[F] \longleftrightarrow P[A, A'] \quad (3.45)$$

which is an isomorphism of vector spaces

$$X_F^c \cong Y_{A,A'}^c, \quad X_F^{c'} \cong Y_{A,A'}^{c'}, \quad X_F^0 \cong Y_{A,A'}^0. \quad (3.46)$$

(ii) *local adjoint-symmetries on $R^q(F)$ and local symmetries on $R^{q+1}(A, A')$, given by*

$$Q[F] \mod D\chi[F] \longleftrightarrow Q[A, A'] \mod A, A', D\chi[A, A'], \quad (3.47)$$

$$Q'[F] \mod D\chi'[F] \longleftrightarrow Q'[A, A'] \mod A, A', D\chi'[A, A'], \quad (3.48)$$

corresponding to the isomorphism of vector spaces

$$\tilde{Y}_F^0 \cong \tilde{X}_{A, A'}^0. \quad (3.49)$$

(iii) *local symmetries on $R^{q+1}(A, A')$ and local symmetries on $R^q(F)$, given by*

$$D(Q[A, A'] \mod D\chi[A, A']) \longleftrightarrow P[F], \quad (3.50)$$

$$D(Q'[A, A'] \mod D\chi'[A, A']) \longleftrightarrow *P[F], \quad (3.51)$$

corresponding to the isomorphism of vector spaces

$$X_{A, A'}^c \cong X_F^c, \quad X_{A, A'}^{c'} \cong X_F^{c'}, \quad \tilde{X}_{A, A'}^0 \cong X_F^0. \quad (3.52)$$

(iv) *local adjoint-symmetries on $R^q(F)$ and local adjoint-symmetries on $R^{q+1}(A, A')$, given by*

$$D(Q[F] \mod D\chi[F]) \longleftrightarrow P[A, A'] \mod F, *F, \quad (3.53)$$

$$D(Q'[F] \mod D\chi'[F]) \longleftrightarrow *P[A, A'] \mod F, *F, \quad (3.54)$$

corresponding to the isomorphism of vector spaces

$$\tilde{Y}_F^0 \cong Y_{A, A'}^0. \quad (3.55)$$

Theorem III.8 (Correspondences related to the systems $DF = D*F = 0$ and $D*DA = 0$)

There is a linear mapping between:

(i) *local symmetries on $R^q(F)$ and local symmetries on $R^{q+1}(A)$, given by*

$$P[F] \mod *F \longleftrightarrow D(Q[A] \mod D\chi[A]), \quad (3.56)$$

corresponding to the isomorphism of vector spaces

$$X_F^c \cong X_A^c, \quad X_F^0 \cong \tilde{X}_A^0. \quad (3.57)$$

(ii) *local adjoint-symmetries of $R^q(F)$ and local symmetries of $R^{q+1}(A)$, given by*

$$Q[F] \bmod D\chi[F] \longleftrightarrow Q[A] \bmod A, D\chi[A], \quad (3.58)$$

$$Q'[F] \bmod D\chi'[F] \longleftrightarrow *'Q[A] \bmod A, D\chi[A], \quad (3.59)$$

corresponding to the isomorphism of vector spaces

$$\tilde{Y}_F^0 \cong \tilde{X}_A^0. \quad (3.60)$$

Proof: We give the proof of Theorem III.5 in detail. The proofs of Theorems III.6–III.8 are readily derived in a similar manner from the decompositions (3.15)–(3.20) combined with the cohomology equations (3.3)–(3.8) and Theorem III.5.

For part (i) of Theorem III.5, by the symmetry decomposition (3.15) with $c = c' = 0$ (i.e. quotienting out the scaling and duality-rotation terms), the pair of differential 1-forms (\tilde{Q}, \tilde{Q}') satisfy the adjoint-symmetry equation on $R^q(F)$. Conversely, in the adjoint-symmetry decomposition (3.16) we see that the differential 2-forms $D\tilde{Q}$, $D\tilde{Q}'$, and their duals are closed due to the adjoint-symmetry equation and hence they directly satisfy the symmetry equations on $R^q(F)$.

Part (ii) follows analogously from the symmetry and adjoint-symmetry decompositions (3.19) and (3.20), including the cases $c \neq 0, c' \neq 0$ via the obvious mappings $F \longleftrightarrow DA$, $*F \longleftrightarrow DA'$.

Finally for part (iii) of the theorem, we apply the 2-form cohomology equation (3.7) to the symmetry equation $D*DQ = 0$ for the differential 1-form Q on $R^q(A)$, which shows that the 2-form $*D(Q + cA)$ is exact for some constant c . Hence the equation $*D(Q + cA) = DQ'$ holds for some differential 1-form Q' on $R^q(A)$, and we see that the dual 2-form $*DQ'$ is exact and therefore is closed. Thus, Q' satisfies the (adjoint-) symmetry equation $D*DQ' = 0$ on $R^q(A)$. \square

We now prove the symmetry and adjoint-symmetry decompositions (3.15)–(3.20).

Proposition III.9 *Every local symmetry $P = P[F]$ of order q of $DF = D*F = 0$ has the form (3.15) for some constants c, c' , where $(\tilde{Q}[F], \tilde{Q}'[F])$ is a local adjoint-symmetry of order $q - 1$ of $DF = D*F = 0$, and conversely.*

Proof: Since $P[F]$ and $*P[F]$ are closed differential 2-forms on $R^q(F)$ for finite q then by the 2-form cohomology equation (3.6) we have

$$P[F] = c_1 F + c_2 *F + D\tilde{Q}[F], \quad (3.61)$$

$$*P[F] = c_3 F + c_4 *F + D\tilde{Q}'[F], \quad (3.62)$$

for some constants c_1, c_2, c_3, c_4 and some differential 1-forms $\tilde{Q}[F], \tilde{Q}'[F]$. Applying $*$ to (3.61) and equating it to (3.62), we obtain

$$(c_1 - c_4)*F - (c_2 + c_3)F = D\tilde{Q}'[F] - *D\tilde{Q}[F]. \quad (3.63)$$

The differential order of the right side is at least one while the left side is of differential order zero, hence on $R^q(F)$ a descent argument (similar to the ones used in the classification results in Ref.[3, 32]) shows that both sides of (3.63) must vanish and so $D\tilde{Q}'[F] = *D\tilde{Q}[F]$. Since $F, *F$ are local 2-form cohomology elements (which are linearly independent), then we must have $c_1 = c_4 \equiv c$, $c_2 = -c_3 \equiv c'$, which establishes (3.15). The converse is immediate. \square

Proposition III.10 *Every local symmetry $Q = Q[A, A']$, $Q' = Q'[A, A']$ of order q of $DA' = *DA$ has the form (3.19) for some constants c, c' , and some scalar functions $\chi[A, A'], \chi'[A, A']$, with $F = DA = -*DA'$, where $(\tilde{Q}[F], \tilde{Q}'[F])$ is a local adjoint-symmetry of order $q - 1$ of $DF = D*F = 0$, and conversely.*

Proof: By the locality-projection theorem, both $DQ = P[F]$, $DQ' = *P[F]$ are local symmetries of $DF = D*F = 0$. Using Proposition III.9 and writing $F = DA, *F = DA'$, we have

$$D(Q[A, A'] - cA - c'A' - \tilde{Q}[F]) = 0, \quad D(Q'[A, A'] - cA' + c'A - \tilde{Q}'[F]) = 0, \quad (3.64)$$

where $(\tilde{Q}[F], \tilde{Q}'[F])$ is a local adjoint-symmetry of $DF = D*F = 0$. Since the local 1-form cohomology on $R^q(A, A')$ for finite q is trivial, we obtain (3.19). The converse is immediate. \square

Proposition III.11 *Every local adjoint-symmetry $P = P[A, A']$ of order q of $DA' = *DA$ has the form (3.20) for some constants c, c' , with $F = DA = -*DA'$, where $(\tilde{Q}[F], \tilde{Q}'[F])$ is a local adjoint-symmetry of order $q - 2$ of $DF = D*F = 0$, and conversely.*

Proof: Since the local 2-form cohomology on $R^q(A, A')$ for finite q is trivial, we have $P = DQ$, $*P = DQ'$ for some differential 1-forms $Q = Q[A, A']$, $Q' = Q'[A, A']$. Hence the pair (Q, Q') is a local symmetry of $DA' = *DA$. By Proposition III.10, D applied to (3.19) yields (3.20). The converse is immediate. \square

Proposition III.12 *Every local (adjoint-) symmetry $Q = Q[A]$ of order q of $D*DA = 0$ has the form (3.17) for some constant c and some scalar function $\chi[A]$, with $F = DA$, where $(\tilde{Q}[F], \tilde{Q}'[F])$ is a local adjoint-symmetry of order $q - 1$ of $DF = D*F = 0$, and conversely.*

Proof: Regard $Q[A]$ as a differential 1-form on $J^q(A, A') \supset J^q(A)$ with no dependence on the coordinates involving A' . Since $DA' = *DA$ implies $D*DA = 0$, the 2-form $*DQ[A]$ is closed on $R^q(A, A')$ whose local 2-form cohomology for finite q is trivial. Hence $*DQ[A] = DQ'[A, A']$ holds for some differential 1-form $Q'[A, A']$ and thus the pair (Q, Q') is a local symmetry of $DA' = *DA$. Then by Proposition III.10, we have

$$Q[A] = cA + c'A' + \tilde{Q}[F] + D\chi[A, A']. \quad (3.65)$$

But since $Q[A]$ is independent of A' we must have $c' = 0$ and $\chi = \chi[A]$. Consequently, we obtain (3.17). The converse is immediate. \square

Finally, we point out the effect of the duality transformation

$$F \rightarrow *F, \quad (A, A') \rightarrow (A', -A) \quad (3.66)$$

on symmetries and adjoint-symmetries of Maxwell's equations and its potential systems, which stems from the explicit classification of adjoint-symmetries $(Q[F], Q'[F])$ on $R^\infty(F)$ given in Ref.[3]. This classification shows that the differential 1-forms $Q[F], Q'[F]$, modulo gradients $D\chi[F], D\chi'[F]$, are *linear* functions of the jet space coordinates and hence have well-defined parity (eigenvalue) decompositions with respect to the combined duality transformations (3.66) and (3.22). Consider more specifically the transformation (3.66) composed with the inverse of the transformation (3.22)

$$(\tilde{Q}[F], \tilde{Q}'[F]) \rightarrow (-\tilde{Q}'[*F], \tilde{Q}[*F]) \quad (3.67)$$

acting on the vector space \tilde{Y}_F^0 . (Here the notation $[*F]$ denotes the duality transformation (3.66) on the jet space coordinates $[F]$ in $J^q(F)$; likewise $[A', -A]$ will denote the duality transformation on the coordinates $[A, A']$ in $J^q(A, A')$. Note that the duality transformation

is well-defined on the corresponding solution jet spaces $R^{q-1}(F)$ and $R^{q-1}(A, A')$. However, it does not exist on either $J^q(A)$ or $R^{q-2}(A)$.) Due to the linearity of $\tilde{Q}[F]$ and $\tilde{Q}'[F]$, the square of the transformation (3.67) is the identity and hence its eigenvalues are ± 1 . The eigenspaces of even/odd parity corresponding to these eigenvalues are then given by

$$\tilde{Q}_\pm[*F] = \pm \tilde{Q}'_\pm[F], \quad \tilde{Q}'_\pm[*F] = \mp \tilde{Q}_\pm[F] \quad \text{on } R^q(F). \quad (3.68)$$

Equivalently, we can write this canonical relation as

$$*\tilde{Q}_\pm[F] = \pm \tilde{Q}_\pm[*F] \quad (3.69)$$

and likewise for $\tilde{Q}'_\pm[F]$, where $*$ is the linear map (3.31). This parity decomposition result (3.68) extends to all solutions of the symmetry equations and adjoint-symmetry equations on $R^q(F)$ and $R^q(A, A')$ through the (cohomology and locality-projection) decompositions in Theorem III.4.

Proposition III.13 *The duality transformation (3.66) on the electromagnetic field and its joint potentials induces a corresponding duality transformation (3.22) on local symmetries and local adjoint-symmetries via*

$$*P_\pm[F] = \pm P_\pm[*F], \quad *P_\pm[A, A'] = \pm P_\pm[A', -A], \quad (3.70)$$

$$Q'_\pm[F] \bmod D\chi'[F] = \pm Q_\pm[*F] \bmod D\chi[F], \quad (3.71)$$

$$Q'_\pm[A, A'] \bmod D\chi'[A, A'] = \pm Q_\pm[A', -A] \bmod D\chi[A, A'], \quad (3.72)$$

where \pm denotes even/odd parity parts with respect to the combined duality transformations

$$P[F] \rightarrow -*P[*F] \quad \text{on } X_F^c \oplus X_F^{c'} \oplus X_F^0, \quad (3.73)$$

$$P[A, A'] \rightarrow -*P[A', -A] \quad \text{on } Y_{A,A'}^c \oplus Y_{A,A'}^{c'} \oplus Y_{A,A'}^0, \quad (3.74)$$

$$(Q[F], Q'[F]) \rightarrow (-Q'[*F], Q[*F]) \quad \text{on } \tilde{Y}_F^0, \quad (3.75)$$

$$(Q[A, A'], Q'[A, A']) \rightarrow (-Q'[A', -A], Q[A', -A]) \quad \text{on } X_{A,A'}^c \oplus X_{A,A'}^{c'} \oplus \tilde{X}_{A,A'}^0. \quad (3.76)$$

In addition we remark that a complete and explicit classification of all local symmetries and adjoint-symmetries of the magnetic and joint potential systems is now available by combining the decompositions in Theorem III.4 with the classification of local adjoint-symmetries of Maxwell's equations from Ref.[3]. An illustration of this result applied to a geometric class of symmetries and adjoint-symmetries will be given later.

C. Conservation law formulas

There is an important application of the preceding results to local conservation laws of Maxwell's equations and its potential systems. We begin with a few preliminaries. In differential form notation a conserved current of order $q < \infty$ given by a vector function Ψ^μ corresponds to a differential 3-form $*\Psi_{\mu\nu\sigma} = \epsilon_{\mu\nu\sigma\tau} \Psi^\tau$ on J^q that is closed on R^∞ . A current is trivial if and only if the corresponding differential 3-form is exact (i.e. $*\Psi_{\mu\nu\sigma} = D_{[\mu} * \Theta_{\nu\sigma]} = \frac{1}{3} \epsilon_{\mu\nu\sigma\tau} D_\alpha \Theta^{\alpha\tau}$) on R^∞ . Thus, the local 3-form cohomology of Maxwell's equations and its potential systems describes the nontrivial conservation laws of these systems. (Note this cohomology is clearly quite rich, in contrast to the local 1-form and 2-form cohomology. See Ref.[30] for a general discussion of conservation laws of field equations from this perspective.) The formulas in Table II that generate all conserved currents (modulo curls) in terms of adjoint-symmetries are summarized in Table IV in this notation.

System	Adjoint-symmetry	Conserved 3-form current
Maxwell's equations $DF = 0$ $D*F = 0$	$DQ'[F] = *DQ[F]$	$*\Psi = \int_0^1 (Q' \wedge F - Q \wedge *F) _{\lambda F} \frac{d\lambda}{\lambda}$
Magnetic potential system $D*DA = 0$	$D*DQ[A] = 0$	$*\Psi = \int_0^1 (A \wedge *DQ - Q \wedge *DA) _{\lambda A} \frac{d\lambda}{\lambda}$
Joint potential system $DA' = *DA$	$DP[A, A'] = 0$ $D*P[A, A'] = 0$	$*\Psi = \int_0^1 (-A \wedge P - A' \wedge *P) _{\lambda A, \lambda A'} \frac{d\lambda}{\lambda}$

TABLE IV: Conserved current formulas in differential form notation

Now, through the adjoint-symmetry decompositions in Theorem III.4, the conserved current formulas for the potential systems can be simplified to remove inessential dependence on the potentials.

Proposition III.14 *For the potential system $D*DA = 0$, all nontrivial local conserved 3-form currents of order q are generated by*

$$*\Psi = \int_0^1 \left(\tilde{Q}' \wedge F - \tilde{Q} \wedge *F \right) \Big|_{\lambda F} \frac{d\lambda}{\lambda} \mod D*\Theta \quad (3.77)$$

for $q \geq 1$, where $\tilde{Q}[F]$ is the gauge-invariant part of the (adjoint-) symmetry $Q[A]$ in the decomposition (3.17) and $\tilde{Q}'[F] = *\tilde{Q}[F]$ is its image under the map (3.31), on $R^{q-1}(F)$.

Proof: We substitute the decomposition (3.17) into the integrand of $*\Psi$ in Table IV to obtain

$$A \wedge *DQ - Q \wedge *DA = A \wedge *D\tilde{Q} - \tilde{Q} \wedge *F + \chi D*F - D(\chi *F), \quad (3.78)$$

where the second last term vanishes on $R(A)$, and the last term is an exact 3-form which we may drop. Next we apply the relation $D*\tilde{Q} = *D\tilde{Q}$ to the first term on the right in (3.78), where, note, $*\tilde{Q}$ is determined only to within a gradient $D\chi'$. This yields

$$A \wedge *D\tilde{Q} = A \wedge D*\tilde{Q} = F \wedge *\tilde{Q} - \chi' DF + D(\chi' F - A \wedge *\tilde{Q}), \quad (3.79)$$

where we again drop the last two terms as before. Hence, the integrand of $*\Psi$ modulo an exact 3-form on $R(A)$ is simply $*\tilde{Q} \wedge F - \tilde{Q} \wedge *F$. \square

Proposition III.15 *For the joint potential system $DA' = *DA$, all nontrivial local conserved 3-form currents of order q are generated by linear combinations of*

$$*\Psi = \int_0^1 \left(-\tilde{Q} \wedge F - \tilde{Q}' \wedge *F \right) \Big|_{\lambda F}^{\frac{d\lambda}{\lambda}} \mod D*\Theta \quad \text{on } R^{q-1}(A, A') \quad (3.80)$$

for $q \geq 1$, and when $q = 1$,

$$*\Psi = -\frac{1}{2}(A \wedge F + A' \wedge *F) \quad \text{on } R(A, A'), \quad (3.81)$$

where $\tilde{Q}[F], \tilde{Q}'[F]$ are determined by the gauge-invariant parts of the adjoint-symmetry $P[A, A']$ and its dual $*P[A, A']$ in the decomposition (3.20), satisfying the relation $\tilde{Q}' = *\tilde{Q}$.

Proof: As in the previous proof, we first substitute the decompositions in (3.20) into the integrand of $*\Psi$ in Table IV, obtaining

$$-A \wedge P - A' \wedge *P = c(-A \wedge F - A' \wedge *F) + c'(-A \wedge *F + A' \wedge F) - A \wedge D\tilde{Q} - A' \wedge D\tilde{Q}'. \quad (3.82)$$

Note that on $R^{q-1}(A, A')$ we have

$$A' \wedge F - A \wedge *F = A' \wedge DA - A \wedge DA' = D(A \wedge A'), \quad (3.83)$$

$$-A \wedge D\tilde{Q} - A' \wedge D\tilde{Q}' = -F \wedge \tilde{Q} - *F \wedge \tilde{Q}' + D(A \wedge \tilde{Q} + A' \wedge \tilde{Q}'). \quad (3.84)$$

The last terms in both (3.83) and (3.84) are exact 3-forms which we drop. Hence, the integrand of $*\Psi$ modulo an exact 3-form on $R^{q-1}(A, A')$ is $c(-A \wedge F - A' \wedge *F) - \tilde{Q} \wedge F - \tilde{Q}' \wedge *F$. \square

These 3-form formulas (3.77), (3.80), (3.81) now lead to our main results pertaining to conservation laws. First, we see that with the exception of one conserved current (3.81) coming from the joint potential system, all other local conserved currents of the two potential systems are equivalent to local conserved currents of Maxwell's equations. In particular, by the adjoint-symmetry decompositions in Theorem III.4, the differential 1-forms $\tilde{Q}[F]$ and $\tilde{Q}'[F]$ in the currents (3.77) and (3.80) can be directly identified with adjoint-symmetries $(Q[F], Q'[F])$ of Maxwell's equations by $Q = \tilde{Q}[F]$ and $Q' = \tilde{Q}'[F]$. There is also a dual identification given by $Q = \tilde{Q}'[F]$, $Q' = -\tilde{Q}[F]$ which reflects the duality invariance (3.22) of the adjoint-symmetry equation $DQ' = *DQ$ on $R^q(F)$. Note that the gauge freedom in the form of the adjoint-symmetries given by

$$Q \rightarrow Q + D\chi[F], \quad Q' \rightarrow Q' + D\chi'[F], \quad (3.85)$$

for arbitrary functions χ, χ' on $J^q(F)$ yields only trivial currents, since

$$D\chi'[F] \wedge F - D\chi[F] \wedge *F = D(\chi'[F]F - \chi[F]*F) \quad \text{on } R(F) \quad (3.86)$$

is an exact 3-form. This result (3.86) applies as well to the exceptional current (3.81), coming from the identification of the adjoint-symmetries $Q = A'$, $Q' = -A$ with an essential dependence on the potentials. Therefore, all conservation laws given by the currents (3.77) and (3.80) as well as the current (3.81) are gauge invariant.

Second, through the mappings relating local adjoint-symmetries of Maxwell's equations to local symmetries of Maxwell's equations and its potential systems in Theorems III.7 and III.8, we obtain a direct correspondence between local conserved currents and local symmetries of each system. This is especially interesting because Maxwell's equations and the joint potential system are not self-adjoint systems, i.e. Noether's theorem relating conservation laws and symmetries through a Lagrangian is inapplicable.

Theorem III.16 *All symmetries of Maxwell's equations induced through its joint potential system directly generate conserved currents via the formula*

$$*\Psi = \int_0^1 (Q' \wedge F - Q \wedge *F)|_{\lambda A, \lambda A'} \frac{d\lambda}{\lambda} = \int_0^1 (A \wedge *P - A' \wedge P)|_{\lambda A, \lambda A'} \frac{d\lambda}{\lambda} \mod D*\Theta \quad (3.87)$$

where the pair of differential 1-forms $(Q[A, A'], Q'[A, A'])$ is identified with any local symmetry \mathbf{X} of the joint potential system, and the differential 2-form $P[A, A']$ is identified with any corresponding symmetry of Maxwell's equations, written in terms of the joint potentials via the relations $DQ[A, A'] = P[A, A']$, $DQ'[A, A'] = *P[A, A']$. For the scaling symmetry (3.23), the current (3.87) is trivial.

The proof of this theorem is similar to that of Proposition III.15 and will be omitted.

We note that this scaling formula (3.87) generates the exceptional current (3.81) naturally from the duality-rotation symmetry (3.24) of the joint potential system and Maxwell's equations. The implications of the explicit dependence on the potentials in this current will be dealt with in more detail in Sec. V. More generally, the scaling formula (3.87) yields conserved currents for any symmetry, *local or nonlocal*, $\mathbf{X} = P \frac{\partial}{\partial F}$ admitted by Maxwell's equations.

Finally, we emphasize a main aspect of the interrelations we have derived between the symmetry structure and the conservation law structure of Maxwell's equations and its potential systems.

Corollary III.17 *No symmetries of $DF = D*F = 0$ with essential dependence on a potential arise from projection of local symmetries of the potential systems $D*DA = 0$ and $DA' = *DA$ under the mapping $F = DA = -*DA'$, due to their gauge freedom $A \rightarrow A + D\chi$, $A' \rightarrow A' + D\chi'$. There is only one conservation law of $DF = D*F = 0$ with explicit dependence on a potential arising from a projection of the local conserved currents of these potential systems, namely, the duality-rotation current (3.81). This exceptional conservation law nevertheless is gauge invariant.*

Thus the locality projection results proven in Ref.[17] for symmetries of well-posed PDE systems when applied to Maxwell's equations extend in a natural sense to gauge-invariant conservation laws. Indeed, a gauge-invariance projection result can be established directly for conservation laws of any locally regular PDE system by an extension of the proof of locality projection for symmetries, which we now outline.

By a locally regular PDE system we mean it and all its prolongations are locally solvable [24] and of constant rank [24], so all conservation laws arise from multipliers. Consider a conserved 3-form current $*\Psi$ of any potential system with gauge freedom. Since conservation of $*\Psi$ holds for all solutions of the potential system, it must remain conserved under

infinitesimal gauge transformations \mathbf{X}_{gauge} on the potentials. Hence the 3-form current $*\bar{\Psi} := \mathbf{X}_{gauge}*\Psi$ is also conserved and consequently projects to a conserved current of the original PDE system through the embedding property of the respective solution spaces [19]. The projected current $*\bar{\Psi}$ necessarily depends on an arbitrary function of x and therefore so does its associated multiplier. Treating this function as an auxiliary dependent variable and applying the corresponding Euler-Lagrange operator, we obtain a divergence identity holding on the PDE system. An integration-by-parts method can then be used to reconstruct an equivalent current from the divergence identity, directly resulting in $*\bar{\Psi} = D*\Theta$ holding for all solutions of the system. Hence we conclude that the 3-form current $*\Psi$ projects to a gauge-invariant conservation law of the given PDE system, namely $*\Psi$ is invariant with respect to the gauge freedom on the potentials modulo trivially conserved (exact 3-form) terms.

D. Geometric symmetries and conservation laws

To conclude this section, we consider the geometric symmetries, adjoint-symmetries, and conserved currents of Maxwell's equations and its potential systems in light of our main results.

Definition III.18 *A differential p -form on J^1 will be called geometric if it is locally constructed from F in the case of Maxwell's equations and from A or A' in the case of the potential systems, using the Minkowski metric, volume form, spacetime coordinates, and exterior derivatives.*

To proceed we will make use of the following Lie derivative identities holding for p -forms ω on any 4-dimensional spacetime manifold: $\mathcal{L}_\xi\omega = \xi \lrcorner d\omega + d(\xi \lrcorner \omega)$, and for $p = 2$, $*\mathcal{L}_\xi\omega = \mathcal{L}_\xi*\omega$. These identities extend in an obvious way to a definition of the Lie derivative on differential forms in a jet space setting, for example, $\mathcal{L}_\xi\omega[F] = \xi \lrcorner D\omega[F] + D(\xi \lrcorner \omega[F])$, and $*\mathcal{L}_\xi\omega[F] = \mathcal{L}_\xi*\omega[F]$. Thus, it follows that any Lie derivatives of F , A , or A' produce geometric differential forms in the sense of Definition III.18.

Accordingly, a symmetry, adjoint-symmetry, or conserved current will be called *geometric* if it is described by a geometric differential form. (Higher-order symmetries, adjoint-symmetries, and conserved currents of an analogous form arise from geometric ones in an

explicit manner by the repeated application of Lie derivatives on F, A, A' , with respect to general conformal Killing vectors, i.e. general conformal symmetry operators.) Note the classes of geometric symmetries, adjoint-symmetries, and conserved currents are each invariant under the duality transformations (3.22) and (3.66).

The geometric symmetries of Maxwell's equations and its potential systems consist of the general conformal symmetries

$$\mathbf{X}_{conf} = \mathcal{L}_\xi F \frac{\partial}{\partial F}, \quad \mathcal{L}_\xi A \frac{\partial}{\partial A}, \quad \mathcal{L}_\xi A \frac{\partial}{\partial A} + \mathcal{L}_\xi A' \frac{\partial}{\partial A'}, \quad (3.88)$$

and their corresponding dual symmetries

$$\mathbf{X}'_{conf} = \mathcal{L}_\xi *F \frac{\partial}{\partial F}, \quad \mathcal{L}_\xi A' \frac{\partial}{\partial A} - \mathcal{L}_\xi A \frac{\partial}{\partial A'}, \quad (3.89)$$

in addition to the obvious scaling and duality-rotation symmetries (3.23) and (3.24). Here ξ denotes a general conformal Killing vector of the Minkowski metric, i.e. $\mathcal{L}_\xi \eta = \Omega \eta$, $\Omega = \frac{1}{2} \text{div } \xi$, representing four translation symmetries, six rotation/boost symmetries, a dilation symmetry, and four conformal (inversion) symmetries. (Note that among these infinitesimal transformations, \mathbf{X}_{scal} , \mathbf{X}_{dual} , \mathbf{X}_{conf} are of point-type [25] while \mathbf{X}'_{conf} is not, i.e. its type is first-order.) It is well known that the symmetries \mathbf{X}_{conf} comprise the 15-dimensional conformal Lie algebra $\mathfrak{so}(4, 2)$, while \mathbf{X}_{conf} and \mathbf{X}'_{conf} together form a 30-dimensional Lie algebra isomorphic to the complexification of $\mathfrak{so}(4, 2)$; the scaling and duality-rotation symmetries \mathbf{X}_{scal} , \mathbf{X}_{dual} commute with both \mathbf{X}_{conf} and \mathbf{X}'_{conf} . Thus we note the Lie algebra of geometric symmetries of Maxwell's equations and its potential systems has the structure $\mathfrak{u}(1)^2 \times \mathfrak{so}(4, 2) \otimes \mathbb{C}$.

The differential 1-forms and 2-forms associated with these symmetries and the corresponding adjoint-symmetries obtained through the mappings in Theorems III.5 to III.8 are summarized in Table V. All these pairs of differential forms $(P, *P)$ and (Q, Q') have even-parity under the respective combined duality transformations (3.73)–(3.76). (We remark that, in contrast, the analogous pairs associated with the chiral symmetries [2, 3] of Maxwell's equations possess odd-parity and exhibit a non-geometric form that involves symmetrized derivatives of F .)

Substitution of the 1-forms and 2-forms in Table V into the conserved current formulas in Table IV generates four currents, two of which are trivial. The two other currents generated are the duality-rotation one (3.81) discussed earlier and one which is equivalent to the well-

Q	Q'	$P = DQ$	$*P = DQ'$
A	A'	F	$*F$
A'	$-A$	$*F$	$-F$
$\xi \lrcorner F$	$\xi \lrcorner *F$	$\mathcal{L}_\xi F$	$\mathcal{L}_\xi *F$
$\xi \lrcorner *F$	$-\xi \lrcorner F$	$\mathcal{L}_\xi *F$	$-\mathcal{L}_\xi F$

TABLE V: 1-forms and 2-forms associated with geometric symmetries and adjoint-symmetries for Maxwell's equations and potential systems

known geometric stress-energy currents of Maxwell's equations

$$\begin{aligned}
*\Psi &= \frac{1}{2}((\xi \lrcorner *F) \wedge F - (\xi \lrcorner F) \wedge *F) \quad \text{on } R(F) \\
&= \frac{1}{2}(A \wedge \mathcal{L}_\xi *F - A' \wedge \mathcal{L}_\xi F) \mod D*\Theta \text{ on } R^1(A, A'),
\end{aligned} \tag{3.90}$$

where ξ is a general conformal Killing vector.

IV. ANALYSIS OF THE JOINT POTENTIAL SYSTEM IN LORENTZ GAUGE

In light of the main results obtained in Sec. III, we must investigate the imposition of gauges in order to obtain new symmetries or new conservation laws of the potential systems for Maxwell's equations. In the case of the magnetic potential system (2.7), some well-known gauges are shown in Table VI. Note that $x^\mu = \{x^0, x^i\}$ denotes time and space coordinates in Minkowski space.

Gauge Name	Description
Lorentz	$\partial^\mu A_\mu(x) = 0$, i.e. $\partial_0 A_0(x) = \partial^i A_i(x)$
Coulomb	$\partial^i A_i(x) = 0$
Temporal	$A_0(x) = 0$
Axial	$n^i A_i(x) = 0$, $n^i = \text{const}$
Cronstrom	$x^\mu A_\mu(x) = 0$, i.e. $x^i A_i(x) = -x^0 A_0(x)$

TABLE VI: Well-known gauges for the magnetic potential system

Since Maxwell's equations (2.3) are manifestly covariant and coordinate-independent, natural gauges to investigate for symmetry purposes should also have these properties. It

should be noted that in 2+1 spacetime dimensions, when temporal gauge or axial gauge is imposed on the magnetic potential system, no new local geometrical symmetries arise [17]; moreover, some of the Poincaré and conformal symmetries are manifestly lost.

A standard covariant gauge choice for the magnetic potential system (2.7) is the Lorentz gauge. Cronstrom's gauge is also covariant, but it is explicitly coordinate-dependent and so we will not consider this gauge here. With Lorentz gauge imposed on the magnetic potential we have the augmented system

$$\partial^\mu \partial_{[\mu} A_{\nu]}(x) = 0, \quad \partial^\mu A_\mu(x) = 0, \quad (4.1)$$

which is no longer self-adjoint, and hence is a non-Lagrangian system. A well-known feature of the potential system (4.1) compared to the system without Lorentz gauge is that the conformal symmetries (3.88) on $A_\mu(x)$ are lost. On the other hand, no new geometric symmetries are gained, as shown in the linear case in Ref.[31].

For the joint potential system (2.10), duality between the electric and magnetic potentials motivates the choice of Lorentz gauge on both $A_\mu(x)$ and $A'_\mu(x)$. Hence, we consider the augmented potential system

$$\partial_{[\mu} A'_{\nu]}(x) = \frac{1}{2} \epsilon_{\mu\nu\sigma\tau} \partial^\sigma A^\tau(x), \quad \partial^\mu A_\mu(x) = 0, \quad \partial^\mu A'_\mu(x) = 0. \quad (4.2)$$

Like the joint potential system itself, the augmented system is not self-adjoint and thus is not a Lagrangian system. Note, importantly, the duality invariance (2.11) on the potentials is retained.

Due to the imposition of Lorentz gauge, the potential system (4.2) becomes a locally well-posed PDE system, with no gauge freedom on the potentials. As is the case for the standard potential system in Lorentz gauge, the potentials still admit a residual freedom given by gradients (2.12) involving scalar functions $\chi(x), \chi'(x)$, but in order to preserve Lorentz gauge these functions are restricted to satisfy the wave equation $\partial^\mu \partial_\mu \chi(x) = 0$, $\partial^\mu \partial_\mu \chi'(x) = 0$. (Hence the transformation (2.12) no longer involves *arbitrary* functions of x^μ .) Modulo this residual freedom, the solutions of the potential system (4.2) in Lorentz gauge are in one-to-one correspondence with the solutions of Maxwell's equations (2.3).

In this section, we will classify geometric symmetries and corresponding conserved currents admitted by this potential system (4.2). We will denote the solution jet space of the system by $\tilde{R}(A, A')$. Note the coordinates of $\tilde{R}(A, A')$ are related to those of $R(A, A')$ by quotienting out the Lorentz gauge equations on derivatives of the potentials.

A. Symmetry analysis

The determining equations for local symmetries $\mathbf{X} = Q_\mu \frac{\partial}{\partial A_\mu} + Q'_\mu \frac{\partial}{\partial A'_\mu}$ of order $q < \infty$ of the potential system (4.2) are given by

$$D_{[\mu} Q'_{\nu]} = *D_{[\mu} Q_{\nu]}, \quad (4.3)$$

$$D^\mu Q_\mu = 0, \quad D^\mu Q'_\mu = 0, \quad (4.4)$$

on $\tilde{R}^q(A, A') \subset J^{q+1}(A, A')$. These equations retain the duality invariance

$$(Q, Q') \longrightarrow (Q', -Q). \quad (4.5)$$

Note that the loss of gauge freedom in this potential system under Lorentz gauge implies there is no gradient freedom in the form of symmetries Q_μ, Q'_μ . We therefore refer to (4.4) as Lorentz gauge equations on (Q, Q') .

Anco & Pohjanpelto have shown that any local symmetry of order $0 \leq q < \infty$ of Maxwell's equations is *linear* in the field $F_{\mu\nu}$ and its derivatives on $R^q(F)$ [2, 32]. The same method can be expected to establish an analogous result for local symmetries of the potential system (4.2). Moreover, the geometric symmetries of Maxwell's equations and the joint potential system have even-parity under the respective combined duality transformations (3.73) and (3.76). This motivates a classification using a linear homogeneous even-parity ansatz for geometric symmetries:

$$Q_\mu[A, A'] = \mathcal{L}_\xi A_\mu + \mathcal{L}_{\xi'} A'_\mu + a_\mu{}^\nu(x) A_\nu + a'_\mu{}^\nu(x) A'_\nu, \quad (4.6)$$

$$Q'_\mu[A, A'] = \mathcal{L}_\xi A'_\mu - \mathcal{L}_{\xi'} A_\mu + a_\mu{}^\nu(x) A'_\nu - a'_\mu{}^\nu(x) A_\nu, \quad (4.7)$$

where $\xi^\nu, \xi'^\nu, a_\mu{}^\nu, a'_\mu{}^\nu$, are functions of x to be determined. Note the even-parity condition characterizing this class of symmetries is expressed by

$$Q'_\mu[A, A'] = Q_\mu[A', -A]. \quad (4.8)$$

Theorem IV.1 *The (even-parity) class of geometric symmetries \mathbf{X} of the form (4.6)–(4.7) admitted by the potential system (4.2) consists of:*

(i) *the scaling and duality-rotation transformations*

$$\mathbf{X}_{scal} = A_\mu \frac{\partial}{\partial A_\mu} + A'_\mu \frac{\partial}{\partial A'_\mu}, \quad (4.9)$$

$$\mathbf{X}_{dual} = A'_\mu \frac{\partial}{\partial A_\mu} - A_\mu \frac{\partial}{\partial A'_\mu}, \quad (4.10)$$

(ii) *the internal rotation/boost transformation*

$$\mathbf{X}_\gamma = \left(\gamma_\mu{}^\nu A_\nu + * \gamma_\mu{}^\nu A'_\nu \right) \frac{\partial}{\partial A_\mu} + \left(\gamma_\mu{}^\nu A'_\nu - * \gamma_\mu{}^\nu A_\nu \right) \frac{\partial}{\partial A'_\mu}, \quad (4.11)$$

with

$$\gamma^{\mu\nu} = \gamma^{[\mu\nu]} = \text{const}, \quad (4.12)$$

(iii) *the homothetic Killing-vector transformation and its dual*

$$\mathbf{X}_{\xi^H} = \mathcal{L}_\xi A_\mu \frac{\partial}{\partial A_\mu} + \mathcal{L}_\xi A'_\mu \frac{\partial}{\partial A'_\mu}, \quad (4.13)$$

$$\mathbf{X}'_{\xi^H} = \mathcal{L}_\xi A'_\mu \frac{\partial}{\partial A_\mu} - \mathcal{L}_\xi A_\mu \frac{\partial}{\partial A'_\mu}, \quad (4.14)$$

with

$$\xi^\mu = k_1{}^\mu + k_2{}^{\mu\nu} x_\nu + k_3 x^\mu, \quad k_1{}^\mu, k_2{}^{\mu\nu} = k_2^{[\mu\nu]}, k_3 = \text{const}, \quad (4.15)$$

(iv) *the conformal Killing-vector transformation and its dual*

$$\mathbf{X}_\xi = \left(\hat{\mathcal{L}}_\xi A_\mu + \zeta_\mu{}^\nu A_\nu + * \zeta_\mu{}^\nu A'_\nu \right) \frac{\partial}{\partial A_\mu} + \left(\hat{\mathcal{L}}_\xi A'_\mu + \zeta_\mu{}^\nu A'_\nu - * \zeta_\mu{}^\nu A_\nu \right) \frac{\partial}{\partial A'_\mu} \quad (4.16)$$

$$\mathbf{X}'_\xi = \left(\hat{\mathcal{L}}_\xi A'_\mu + \zeta_\mu{}^\nu A'_\nu - * \zeta_\mu{}^\nu A_\nu \right) \frac{\partial}{\partial A_\mu} - \left(\hat{\mathcal{L}}_\xi A_\mu + \zeta_\mu{}^\nu A_\nu + * \zeta_\mu{}^\nu A'_\nu \right) \frac{\partial}{\partial A'_\mu} \quad (4.17)$$

where $\hat{\mathcal{L}}_\xi := \mathcal{L}_\xi + \frac{1}{4}\Omega$ and $\zeta^{\mu\nu} := -\frac{1}{2}\partial^{[\mu}\xi^{\nu]}$, $\Omega := \frac{1}{2}\partial_\mu\xi^\mu$, with

$$\xi^\mu = k_4{}^\sigma x_\sigma x^\mu - \frac{1}{2}k_4{}^\mu x^\sigma x_\sigma, \quad k_4{}^\mu = \text{const}. \quad (4.18)$$

This class is closed under the duality transformation (4.5).

Before outlining the proof of this theorem, we first discuss geometrical features of these symmetries and the structure of their Lie algebra.

The parameters $k_1{}^\mu$, $k_2{}^{\mu\nu}$, k_3 , $k_4{}^\mu$ appearing in the homothetic/conformal Killing vectors ξ^μ correspond respectively to four translations, three rotations and three boosts, one dilation, and four inversions of M^4 , which are conformal isometries determined by the conformal Killing equation

$$\mathcal{L}_\xi \eta_{\mu\nu} = \Omega \eta_{\mu\nu}. \quad (4.19)$$

The isometries (4.19) generated by translations, rotations/boosts, and a dilation comprise the homothetic Killing vectors (4.15) on M^4 and induce a corresponding Lie derivative action on the potentials, i.e. infinitesimal Poincaré and dilation transformations. In contrast to the situation for the standard potential system in Lorentz gauge (4.1), the infinitesimal conformal transformation associated with genuine conformal isometries (4.19) given by inversions on M^4 are not lost. However, the transformation is modified through combining a weighted Lie derivative with respect to a conformal Killing vector (4.18) and an internal rotation on the potentials via the coefficients $\zeta^{\mu\nu}$ and $*\zeta^{\mu\nu}$.

The infinitesimal internal rotation/boost transformation with parameters $\gamma^{\mu\nu} = \gamma^{[\mu\nu]}$ is genuinely new as there is no local counterpart of it in Maxwell's equations or the standard potential system with or without Lorentz gauge. Along with the infinitesimal scaling and duality-rotation transformations, these transformations are internal (non-spacetime) symmetries in the sense that there is no associated motion on spacetime. Note they comprise three internal rotations and three internal boosts in addition to the one scaling and one duality-rotation.

We give an alternative representation for the internal symmetries as follows. At any point in spacetime, identify $\mathbb{R}^4 \times \mathbb{R}^4 \cong \mathbb{R}^4 \otimes \mathbb{R}^2$, and introduce basis elements e_1, e_2 for \mathbb{R}^2 . We will identify the jet space coordinates of the potentials A_μ, A'_μ with $A_\mu \otimes e_1, A_\mu \otimes e_2$. Let $\text{id} \in \text{Hom}(\mathbb{R}^4)$ be the identity map, and $R \in \text{Hom}(\mathbb{R}^2)$ a standard rotation on \mathbb{R}^2 , i.e. $\text{id}(v_\mu) = v_\mu$, $R(e_1) = e_2$, $R(e_2) = -e_1$, for $v_\mu \in \mathbb{R}^4$. Then the duality-rotation is given by the transformation

$$\text{id} \otimes R \in \text{Hom}(\mathbb{R}^4 \otimes \mathbb{R}^2) \quad (4.20)$$

which acts non-trivially only on the \mathbb{R}^2 factor, or off-diagonally on the space $\mathbb{R}^4 \otimes \mathbb{R}^2$, with matrix representation

$$\left(\begin{array}{c|c} 0 & -\mathbb{I} \\ \hline \mathbb{I} & 0 \end{array} \right). \quad (4.21)$$

The internal rotations/boosts combine a nontrivial action on the \mathbb{R}^2 and \mathbb{R}^4 factors. Define $R_\gamma \in \text{Hom}(\mathbb{R}^4)$ to be the standard rotation/boost operator with parameters $\gamma^{\mu\nu} = \gamma^{[\mu\nu]}$. Recall that the rotations/boosts act infinitesimally by $R_\gamma(v_\mu) = \gamma_\mu{}^\nu v_\nu$ for $v_\mu \in \mathbb{R}^4$. Here the constant parameters $\gamma^{\mu\nu}$ geometrically determine a 2-dimensional plane in \mathbb{R}^4 on which the rotation/boost takes place (namely, the cokernel of R_γ). With this notation, the internal

rotations/boosts act via

$$R_\gamma \otimes \text{id} + R_{*\gamma} \otimes R \in \text{Hom}(\mathbb{R}^4 \otimes \mathbb{R}^2). \quad (4.22)$$

We then see that the transformation (4.22) is a sum of diagonal and off-diagonal rotations/boosts on $\mathbb{R}^4 \otimes \mathbb{R}^2$: the diagonal action involves a standard rotation/boost on \mathbb{R}^4 , and the plane for the off-diagonal rotation/boost is the dual of the plane for the diagonal rotation/boost.

The geometric symmetries (4.9)–(4.17) comprise a 38-dimensional symmetry algebra. To describe its commutator structure, we first extend the transformation \mathbf{X}_ξ and its dual \mathbf{X}'_ξ to general conformal Killing vectors (4.19).

Proposition IV.2 *For a homothetic Killing vector $\xi^\mu = k_1^\mu + k_2^{\mu\nu} x_\nu + k_3 x^\mu$,*

$$\mathbf{X}_\xi = \mathbf{X}_{\xi^H} + \mathbf{X}_\gamma + \frac{1}{4}\Omega \mathbf{X}_{scal}, \quad \mathbf{X}'_\xi = \mathbf{X}'_{\xi^H} - \mathbf{X}_{*\gamma} + \frac{1}{4}\Omega \mathbf{X}_{dual}, \quad \gamma = \zeta \quad (4.23)$$

is a geometric symmetry of the joint potential system in Lorentz gauge, where $\zeta^{\mu\nu} = \frac{1}{2}k_2^{\mu\nu} = \text{const}$, $\Omega = 2k_3 = \text{const}$ are the (scaled) curl and divergence of ξ^μ .

Let $[\gamma_1, \gamma_2]^{\mu\nu} = 2\gamma_1^{\sigma[\mu} \gamma_2^{\nu]}_\sigma$ denote the commutator of two skew-tensors $\gamma_1^{\mu\nu}, \gamma_2^{\mu\nu}$ viewed as matrices, and $[\xi_1, \xi_2]^\mu = \mathcal{L}_{\xi_1} \xi_2^\mu = -\mathcal{L}_{\xi_2} \xi_1^\mu$ denote the commutator of two general conformal Killing vectors ξ_1^μ, ξ_2^μ . Note a general conformal Killing vector ξ^μ has a decomposition into a sum of a homothetic Killing vector (4.15) and a conformal Killing vector (4.18) whose curl and divergence are given by

$$\zeta^{\mu\nu} = -\frac{1}{2}\partial^{[\mu} \xi^{\nu]} = \frac{1}{2}k_2^{\mu\nu} - k_4^{[\mu} x^{\nu]}, \quad \Omega = \frac{1}{2}\partial_\mu \xi^\mu = 2k_3 + 2k_4^\mu x_\mu, \quad (4.24)$$

where $\zeta^{\mu\nu}$ and Ω are constant only in the case of homothetic Killing vector. Recall, any two conformal Killing vectors (4.18) commute, while the commutator of any two homothetic Killing vectors (4.15) is again a homothetic Killing vector, which is given by the well-known Poincaré Lie algebra [10, 11, 12] enlarged by the one-dimensional Lie algebra of dilations. (In particular, dilations commute with all homothetic Killing vectors except translations, whose commutator is again a translation i.e. $[x, k_1]^\mu = -k_1^\mu$; the Poincaré Lie algebra is the semidirect product of the Lie algebra $\mathfrak{so}(3,1)$ of rotations/boosts and the abelian Lie algebra $\mathfrak{u}(1)^4 \cong \mathbb{R}^4$ of translations.) The commutator of a conformal Killing vector (4.18) and a homothetic Killing vector (4.15) is given by the parameters

$$\bar{k}_1^\mu = 0, \quad \bar{k}_2^{\mu\nu} = 2k_4^{[\mu} k_1^{\nu]}, \quad \bar{k}_3 = -k_{4\nu} k_1^\nu, \quad \bar{k}_4^\mu = -k_4^\mu k_3 - k_{4\nu} k_2^{\nu\mu}. \quad (4.25)$$

Theorem IV.3 (Geometric symmetry algebra)

The nonzero commutators of the geometric symmetries (4.9)–(4.17) of the joint potential system in Lorentz gauge have the algebraic structure

$$[\mathbf{X}_{\gamma_1}, \mathbf{X}_{\gamma_2}] = 2\mathbf{X}_{\gamma_3}, \quad \text{where } \gamma_3 = [\gamma_2, \gamma_1], \quad (4.26)$$

$$[\mathbf{X}_{\xi_1}, \mathbf{X}_{\xi_2}] = -[\mathbf{X}'_{\xi_1}, \mathbf{X}'_{\xi_2}] = \mathbf{X}_{\xi_3}, \quad [\mathbf{X}_{\xi_1}, \mathbf{X}'_{\xi_2}] = \mathbf{X}'_{\xi_3}, \quad \text{where } \xi_3 = [\xi_2, \xi_1]. \quad (4.27)$$

Moreover, $\mathbf{X}_{scal}, \mathbf{X}_{dual}$ span a two-dimensional abelian Lie algebra, \mathbf{X}_γ spans the six-dimensional rotation/boost Lie algebra $\mathfrak{so}(3, 1)$, \mathbf{X}_ξ spans the 15-dimensional conformal Lie algebra $\mathfrak{so}(4, 2)$; the Lie algebra spanned by $\mathbf{X}_\xi, \mathbf{X}'_\xi$ is isomorphic to the complexification of $\mathfrak{so}(4, 2)$. Thus the geometric symmetries together form a 38-dimensional Lie algebra $\mathfrak{u}(1)^2 \times \mathfrak{so}(3, 1) \times \mathfrak{so}(4, 2) \otimes \mathbb{C}$.

The computation of this symmetry algebra is straightforward and will be omitted. To conclude the symmetry analysis, we now derive the geometric symmetry classification by solving the symmetry determining equations (4.3) and (4.4).

Proof of Theorem IV.1: Explicit coordinates for the solution jet space $\tilde{R}(A, A')$ are given by $(x^\mu, A_\nu, A'_\nu, F_{\mu\nu}, \text{trfr} A_{\mu\nu}, \text{trfr} A'_{\mu\nu})$ where

$$F_{\mu\nu} = A_{[\nu, \mu]} = -\frac{1}{2}\epsilon_{\mu\nu}^{\sigma\tau} A'_{\tau, \sigma}, \quad A_{\mu\nu} = A_{(\nu, \mu)}, \quad A'_{\mu\nu} = A'_{(\nu, \mu)}. \quad (4.28)$$

These components represent the linearly independent parts of the potentials and their first-order derivatives at a point in spacetime, subject to the system equations (4.2).

To proceed, we substitute (4.6) and (4.7) into the determining equations (4.3) and (4.4). The second-order derivative terms in A and A' are found to vanish on $\tilde{R}^1(A, A') \subset J^2(A, A')$. For the remaining terms, we extract the coefficients of the linearly independent coordinates on $\tilde{R}(A, A')$ as follows: (i) A_ν, A'_ν , (ii) $F_{\mu\nu}$, (iii) $\text{trfr} A_{\mu\nu}, \text{trfr} A'_{\mu\nu}$. By setting the coefficients to vanish, we obtain the equations

$$\partial_{[\alpha} a_{\beta]\sigma} = \frac{1}{2}\epsilon_{\alpha\beta}^{\mu\nu} \partial_\mu a'_{\nu\sigma}, \quad (4.29)$$

$$\partial_\sigma \bar{a}^{\sigma\alpha} = 0, \quad \partial_\sigma \bar{a}'^{\sigma\alpha} = 0, \quad (4.30)$$

$$a_{[\alpha}^{(\mu} \delta^{\nu)}_{\beta]} + \frac{1}{2} a'^{\sigma(\mu} \epsilon^{\nu)}_{\alpha\beta\sigma} = \frac{1}{4} \eta^{\mu\nu} (a_{[\alpha\beta]} - \frac{1}{2} \epsilon_{\alpha\beta\sigma\tau} a'^{\sigma\tau}), \quad (4.31)$$

$$\tilde{a}_{(\alpha\beta)} = \frac{1}{4} \eta_{\alpha\beta} \tilde{a}_\sigma{}^\sigma, \quad \tilde{a}'_{(\alpha\beta)} = \frac{1}{4} \eta_{\alpha\beta} \tilde{a}'_\sigma{}^\sigma, \quad (4.32)$$

$$\tilde{a}_{[\alpha}{}^\sigma \epsilon_{\beta]\sigma}{}^{\mu\nu} - \tilde{a}^{\sigma[\mu} \epsilon^{\nu]}_{\alpha\beta\sigma} + 2(\tilde{a}'_{[\alpha}{}^{[\mu} + \tilde{a}'^{[\mu}{}_{[\alpha} \delta^{\nu]}_{\beta]}) \delta^{\sigma]}_{\sigma} - \tilde{a}'_\sigma{}^\sigma \delta^{[\mu}{}_\alpha \delta^{\nu]}_{\beta]} = 0, \quad (4.33)$$

$$a'_{[\alpha\beta]} = \frac{1}{2} \epsilon_{\alpha\beta\sigma\tau} a^{\sigma\tau}, \quad (4.34)$$

where, for notational convenience, we have defined

$$\bar{a}_{\alpha\beta} := a_{\alpha\beta} + \partial_\alpha \xi_\beta, \quad \bar{a}'_{\alpha\beta} := a'_{\alpha\beta} + \partial_\alpha \xi'_\beta, \quad (4.35)$$

$$\tilde{a}_{\alpha\beta} := a_{\alpha\beta} + 2\partial_\alpha \xi_\beta, \quad \tilde{a}'_{\alpha\beta} := a'_{\alpha\beta} + 2\partial_\alpha \xi'_\beta. \quad (4.36)$$

From (4.31), contracting on ν, β , and then symmetrizing on α, μ , we find

$$a_{(\alpha\beta)} = \frac{1}{4}\eta_{\alpha\beta}a_\sigma{}^\sigma. \quad (4.37)$$

Similarly, multiplication of (4.31) by $\epsilon^{\alpha\beta\sigma\tau}$ followed by a similar contraction and symmetrization leads to

$$a'_{(\alpha\beta)} = \frac{1}{4}\eta_{\alpha\beta}a'_\sigma{}^\sigma. \quad (4.38)$$

Combining (4.37) and (4.38) with (4.32), we obtain

$$\partial_{(\alpha}\xi_{\beta)} = \frac{1}{4}\eta_{\alpha\beta}\partial_\sigma\xi^\sigma, \quad \partial_{(\alpha}\xi'_{\beta)} = \frac{1}{4}\eta_{\alpha\beta}\partial_\sigma\xi'^\sigma. \quad (4.39)$$

Through (4.37)–(4.39) and (4.34), the equations (4.31)–(4.33) reduce to identities.

The differential equation (4.39) is equivalent to the conformal Killing equation (4.19) and hence both $\xi^\mu(x)$ and $\xi'^\mu(x)$ are general conformal Killing vectors:

$$\xi^\mu = k_1{}^\mu + k_2{}^{\mu\nu}x_\nu + k_3x^\mu + k_4{}^\nu x_\nu x^\mu - \frac{1}{2}k_4{}^\mu x^\nu x_\nu, \quad (4.40)$$

$$\xi'^\mu = k'_1{}^\mu + k'_2{}^{\mu\nu}x_\nu + k'_3x^\mu + k'_4{}^\nu x_\nu x^\mu - \frac{1}{2}k'_4{}^\mu x^\nu x_\nu, \quad (4.41)$$

where $k_1{}^\mu$, $k_2{}^{\mu\nu} = k_2^{[\mu\nu]}$, k_3 , $k_4{}^\mu$, and $k'_1{}^\mu$, $k'_2{}^{\mu\nu} = k'_2^{[\mu\nu]}$, k'_3 , $k'_4{}^\mu$ are constants.

We decompose $a_{\nu\sigma}$ and $a'_{\nu\sigma}$ into their symmetric and antisymmetric parts using (4.37) and (4.38), with the notation

$$\hat{a}_{\nu\sigma} := a_{[\nu\sigma]}, \quad \hat{a}'_{\nu\sigma} := a'_{[\nu\sigma]}. \quad (4.42)$$

Elimination of $\hat{a}'_{\nu\sigma}$ in (4.29) using (4.34) then gives the equation

$$\partial_{[\alpha}\hat{a}_{\beta]\sigma} - \frac{1}{2}\partial_\sigma\hat{a}_{\alpha\beta} = \eta_{\sigma[\alpha}\partial^\mu\hat{a}_{\beta]\mu} + \frac{1}{4}\eta_{\sigma[\alpha}\partial_{\beta]}a_\nu{}^\nu + \frac{1}{8}\epsilon_{\alpha\beta\mu\sigma}\partial^\mu a'_\nu{}^\nu. \quad (4.43)$$

By contracting (4.43) on β, σ , followed by using (4.30) and (4.40), we obtain

$$\partial_\alpha a_\sigma{}^\sigma = \frac{4}{3}\partial^\sigma\hat{a}_{\sigma\alpha} = -\partial^\sigma\partial_\sigma\xi_\alpha = 2k_{4\alpha}. \quad (4.44)$$

Consequently, the trace-part of $a_{\alpha\beta}$ is

$$a_\sigma{}^\sigma = 4\lambda + 2k_4{}^\sigma x_\sigma, \quad (4.45)$$

where λ is a constant. Then, a similar elimination of $\hat{a}_{\nu\sigma}$ in (4.29) leads to the trace-part of $a'_{\alpha\beta}$,

$$a'_{\sigma}{}^{\sigma} = 4\lambda' + 2k_4'^{\sigma} x_{\sigma}, \quad (4.46)$$

where λ' is a constant.

Hence (4.43) becomes

$$\partial_{[\alpha} \hat{a}_{\beta]\sigma} - \frac{1}{2} \partial_{\sigma} \hat{a}_{\alpha\beta} = k_{4[\alpha} \eta_{\beta]\sigma} - \frac{1}{4} \epsilon_{\alpha\beta\sigma\tau} k_4'^{\tau} \quad (4.47)$$

through (4.44)–(4.46). Antisymmetrization of (4.47) on α, β, σ leads to

$$\partial_{[\alpha} \hat{a}_{\beta]\sigma} + \frac{1}{2} \partial_{\sigma} \hat{a}_{\alpha\beta} = -\frac{3}{4} \epsilon_{\alpha\beta\sigma\tau} k_4'^{\tau}. \quad (4.48)$$

Combining (4.47) and (4.48) we obtain

$$\partial_{\sigma} \hat{a}_{\alpha\beta} = -k_{4[\alpha} \eta_{\beta]\sigma} - \frac{1}{2} \epsilon_{\alpha\beta\sigma\tau} k_4'^{\tau}, \quad (4.49)$$

which yields

$$\hat{a}_{\alpha\beta} = \gamma_{\alpha\beta} + \zeta_{\alpha\beta} - *\zeta'_{\alpha\beta}, \quad (4.50)$$

where $\gamma_{\alpha\beta} = \gamma_{[\alpha\beta]} = \text{const}$, and $\zeta_{\alpha\beta} = -\frac{1}{2} \partial_{[\alpha} \xi_{\beta]} = -k_{4[\alpha} x_{\beta]}$, $\zeta'_{\alpha\beta} = -\frac{1}{2} \partial_{[\alpha} \xi'_{\beta]} = -k'_{4[\alpha} x_{\beta]}$. Finally, using (4.34) we have

$$\hat{a}'_{\alpha\beta} = *\gamma_{\alpha\beta} + *\zeta_{\alpha\beta} + \zeta'_{\alpha\beta}. \quad (4.51)$$

All equations (4.29)–(4.34) now reduce to identities. \square

We remark that if the ansatz (4.6) and (4.7) is generalized to include odd parity terms of zeroth order on $J^1(A, A')$, then the proof goes through and no new symmetries are obtained. It would be a natural generalization to next include odd parity first-order terms, but we have not investigated the outcome.

B. Conservation law analysis

Since the solution space of the joint potential system in Lorentz gauge has a natural embedding into the solution space of the unaugmented potential system, both the duality-rotation current (3.81) connected with duality rotations on the potentials and the stress-energy currents (3.90) associated with general conformal Killing vectors, discussed in

Sec. III D, continue to be admitted when Lorentz gauge is imposed. New conserved currents are suggested by the appearance of the new local symmetries admitted under Lorentz gauge.

The joint potential system remains non-self-adjoint in Lorentz gauge and hence it is a non-Lagrangian system. Consequently, conserved currents arise through adjoint-symmetries via a scaling formula similar to the one in Table II for the system without Lorentz gauge.

Local adjoint-symmetries of system (4.2) consist of a differential 2-form $P_{\mu\nu}[A, A']$ as in the case without Lorentz gauge, plus a pair of differential scalar functions $\chi[A, A'], \chi'[A, A']$, which arise respectively from the adjoint of the symmetry equations (4.3) and (4.4). In particular, the determining equations for local adjoint-symmetries of order $q < \infty$ take the form

$$D^\mu P_{\mu\nu} + D_\nu \chi' = 0, \quad (4.52)$$

$$D^\mu *P_{\mu\nu} - D_\nu \chi = 0, \quad (4.53)$$

on $R^q(A, A')$. Note these equations have the duality invariance

$$(P_{\mu\nu}, \chi, \chi') \rightarrow (*P_{\mu\nu}, \chi', -\chi). \quad (4.54)$$

Proposition IV.4 *All nontrivial local conserved currents of the potential system (4.2) are generated from local adjoint-symmetries by the conserved current formula*

$$\Psi^\mu = \int_0^1 \left(P^{\mu\nu} A'_\nu - *P^{\mu\nu} A_\nu + \chi A^\mu + \chi' A'^\mu \right) \Big|_{\lambda A, \lambda A'} \frac{d\lambda}{\lambda}. \quad (4.55)$$

To apply this result, rather than solve the adjoint-symmetry equations we will now utilize our symmetry classification results and derive a mapping from local symmetries to local adjoint-symmetries, extending the mapping (3.35)–(3.36) obtained in Theorem III.5 for the joint potential system without gauges.

Since this map (3.35)–(3.36) relied only on the adjoint relation between the respective determining equations for local symmetries $Q_\mu[A, A'], Q'_\mu[A, A']$ and local adjoint-symmetries $P_{\mu\nu}[A, A']$ of the unaugmented potential system, it carries over to the potential system in Lorentz gauge if we project out the functions $\chi[A, A'], \chi'[A, A']$ associated with the Lorentz gauge equations (4.4).

Theorem IV.5 *For the joint potential system in Lorentz gauge, there is a linear mapping from local symmetries into local adjoint-symmetries given by*

$$(Q_\nu, Q'_\nu) \longrightarrow (P_{\mu\nu} = D_{[\mu} Q_{\nu]}, *P_{\mu\nu} = D_{[\mu} Q'_{\nu]}, \chi = 0, \chi' = 0) \quad (4.56)$$

as well as a dual mapping

$$(Q_\nu, Q'_\nu) \longrightarrow (P_{\mu\nu} = -D_{[\mu} Q'_{\nu]}, *P_{\mu\nu} = D_{[\mu} Q_{\nu]}, \chi = 0, \chi' = 0) \quad (4.57)$$

coming from the duality invariance (4.54). Associated to these correspondences (4.56) and (4.57) are respective formulas (4.55) that directly generate local conserved currents

$$\Psi^\mu = \int_0^1 (Q'_\nu F^{\mu\nu} - Q_\nu *F^{\mu\nu}) \Big|_{\lambda A, \lambda A'} \frac{d\lambda}{\lambda} \quad (4.58)$$

and

$$\Psi^\mu = \int_0^1 (Q_\nu F^{\mu\nu} + Q'_\nu *F^{\mu\nu}) \Big|_{\lambda A, \lambda A'} \frac{d\lambda}{\lambda} \quad (4.59)$$

from local symmetries of this potential system.

Proof: The adjoint-symmetry equations (4.52) and (4.53) reduce directly to the symmetry equations (4.3) and (4.4) through substitution of the mappings (4.56) and (4.57). Similarly, the conserved current formula (4.55) reduces to formulas (4.58) and (4.59) modulo curls, since $A_\nu D^{[\mu} Q^{\nu]} = A_\nu *D^{[\mu} Q^{\nu]} = \frac{1}{2}\epsilon^{\mu\nu\alpha\beta} (Q_\beta D_\nu A_\alpha + D_\alpha (Q_\beta A_\nu))$ and likewise for the other term $A'_\nu D^{[\mu} Q^{\nu]} = -A'_\nu *D^{[\mu} Q^{\nu]}$. \square

We remark that a converse for either correspondence (4.56) or (4.57) would rely on generalizing the local 2-form cohomology theorem to the joint potential system in Lorentz gauge, which is unnecessary for the purpose of generating conserved currents.

Of the two conserved current formulas, the latter one (4.59) is distinguished by the property that it generates a trivial current from the scaling symmetry (4.9) and a nontrivial current from the duality-rotation symmetry (4.10) (while this correspondence is reversed by the dual formula (4.58)). We emphasize that both formulas generate the same conserved currents when applied to the class of geometric symmetries (4.9)–(4.17), since this class exhibits duality-invariance (4.5).

We now list in Table VII the conserved currents that arise through formula (4.59) from the geometric symmetries (4.9)–(4.17) classified in Theorem IV.1. For comparison, we write out the stress-energy currents,

$$\Psi^\mu = \xi^\sigma (F_{\sigma\nu} F^{\mu\nu} + *F_{\sigma\nu} *F^{\mu\nu}) := T^\mu{}_\sigma \xi^\sigma \quad (4.60)$$

where ξ^σ is a general conformal Killing vector and $T^\mu{}_\sigma$ denotes the conserved stress-energy tensor of Maxwell's equations.

	Geometric Symmetry	Conserved current
scaling	$Q_\mu = A_\mu$ $Q'_\mu = A'_\mu$	$\Psi^\mu_{scal} = \frac{1}{2}(A_\nu F^{\mu\nu} + A'_\nu *F^{\mu\nu})$ $= D_\nu \left(\frac{1}{4} \epsilon^{\mu\nu\sigma\tau} A_\sigma A'_\tau \right)$
duality-rotation	$Q_\mu = A'_\mu$ $Q'_\mu = -A_\mu$	$\Psi^\mu_{dual} = \frac{1}{2}(A'_\nu F^{\mu\nu} - A_\nu *F^{\mu\nu})$
internal rotations/boosts $\gamma_{\mu\nu}$ given by (4.12)	$Q_\mu = \gamma_\mu^\nu A_\nu + *\gamma_\mu^\nu A'_\nu$ $Q'_\mu = \gamma_\mu^\nu A'_\nu - *\gamma_\mu^\nu A_\nu$	$\Psi^\mu_\gamma = \frac{1}{2} \gamma_\nu^\sigma (A_\sigma F^{\mu\nu} + A'_\sigma *F^{\mu\nu})$ $+ \frac{1}{2} *\gamma_\nu^\sigma (A'_\sigma F^{\mu\nu} - A_\sigma *F^{\mu\nu})$
general conformal Killing vector $\xi^\mu, \zeta^{\mu\nu}$ given by (4.18)	$Q_\mu = \hat{\mathcal{L}}_\xi A_\mu + \zeta_\mu^\nu A_\nu + *\zeta_\mu^\nu A'_\nu$ $Q'_\mu = \hat{\mathcal{L}}_\xi A'_\mu + \zeta_\mu^\nu A'_\nu - *\zeta_\mu^\nu A_\nu$	$\Psi^\mu_\xi = \frac{1}{2}(\hat{\mathcal{L}}_\xi A_\nu)F^{\mu\nu} + \frac{1}{2}(\hat{\mathcal{L}}_\xi A'_\nu)*F^{\mu\nu}$ $+ \frac{1}{2}\zeta_\nu^\sigma (A_\sigma F^{\mu\nu} + A'_\sigma *F^{\mu\nu})$ $+ \frac{1}{2}*\zeta_\nu^\sigma (A'_\sigma F^{\mu\nu} - A_\sigma *F^{\mu\nu})$
general conformal Killing vector dual	$Q_\mu = \hat{\mathcal{L}}_\xi A'_\mu + \zeta_\mu^\nu A'_\nu - *\zeta_\mu^\nu A_\nu$ $Q'_\mu = -\hat{\mathcal{L}}_\xi A_\mu - \zeta_\mu^\nu A_\nu - *\zeta_\mu^\nu A'_\nu$	$\Psi'^\mu_\xi = \frac{1}{2}(\hat{\mathcal{L}}_\xi A'_\nu)F^{\mu\nu} - \frac{1}{2}(\hat{\mathcal{L}}_\xi A_\nu)*F^{\mu\nu}$ $+ \frac{1}{2}\zeta_\nu^\sigma (A'_\sigma F^{\mu\nu} - A_\sigma *F^{\mu\nu})$ $- \frac{1}{2}*\zeta_\nu^\sigma (A_\sigma F^{\mu\nu} + A'_\sigma *F^{\mu\nu})$
Lie derivative terms in Q_μ, Q'_μ $\hat{\mathcal{L}}_\xi = \mathcal{L}_\xi + \frac{1}{4}\Omega$ Ω given by (4.18)		$\frac{1}{2}(\mathcal{L}_\xi A_\nu)F^{\mu\nu} + \frac{1}{2}(\mathcal{L}_\xi A'_\nu)*F^{\mu\nu}$ $= D_\nu \left(\frac{1}{2}\xi^\sigma (A_\sigma F^{\mu\nu} + A'_\sigma *F^{\mu\nu}) \right)$ $+ T^\mu_\sigma \xi^\sigma$ <hr/> $\frac{1}{2}(\mathcal{L}_\xi A'_\nu)F^{\mu\nu} - \frac{1}{2}(\mathcal{L}_\xi A_\nu)*F^{\mu\nu}$ $= D_\nu \left(\frac{1}{2}\xi^\sigma (A'_\sigma F^{\mu\nu} - A_\sigma *F^{\mu\nu}) \right)$

TABLE VII: Conserved currents derived from geometric symmetries of the joint potential system in Lorentz gauge

Trivial currents (i.e. curls) are produced by both the scaling symmetry (4.9) and the homothetic Killing-vector dual symmetries (4.14). The homothetic Killing-vector symmetries (4.13) themselves reproduce the same stress-energy currents (modulo trivial currents) as admitted by the unaugmented potential system.

In contrast the currents associated with the new internal rotation/boost symmetries (4.11) as well as the conformal Killing-vector symmetries (4.16) and their dual symmetries (4.17) are genuinely new local conserved currents. Nontriviality of these currents is established in Sec. V, where we will simplify all inessential dependence on potentials through the embedding

of the solution space of the potential system (4.2) into the solution space of Maxwell's equations.

Finally, we remark that the duality-rotation current, internal rotation/boost currents, conformal Killing-vector currents and their dual currents are related by considering a general conformal Killing vector as was noted for the corresponding symmetries in Proposition IV.2.

Proposition IV.6 *For a homothetic Killing vector $\xi^\mu = k_1^\mu + k_2^{\mu\nu}x_\nu + k_3x^\mu$,*

$$\Psi_\xi^\mu = T^\mu{}_\sigma \xi^\sigma + \Psi_\gamma^\mu \text{ mod curls}, \quad \Psi_\xi^{\prime\mu} = -\Psi_{*\gamma}^\mu + \frac{1}{4}\Omega\Psi_{dual}^\mu \text{ mod curls}, \quad \gamma = \zeta \quad (4.61)$$

is a geometric current of the joint potential system in Lorentz gauge, where $\zeta^{\mu\nu} = \frac{1}{2}k_2^{\mu\nu} = \text{const}$, $\Omega = 2k_3 = \text{const}$.

V. NEW NONLOCAL SYMMETRIES AND CONSERVATION LAWS OF MAXWELL'S EQUATIONS

In this section we derive nonlocal symmetries and nonlocal conserved currents of Maxwell's equations through projection of the new geometric symmetries and new geometric currents of the joint potential system in Lorentz gauge found in Sec. IV. The projection is defined on $J^1(A, A')$ using the natural embedding of the solution jet space $\tilde{R}(A, A')$ of the potentials in Lorentz gauge into the solution jet space of Maxwell's equations as follows: We note first that, with the introduction of potentials, points in $J^q(F)$ are identified with equivalence classes of points in $J^{q+1}(A, A')$ under (prolonged) gauge transformations $A_\mu \rightarrow A_\mu + D_\mu\chi$, $A'_\mu \rightarrow A'_\mu + D_\mu\chi'$, for any functions χ, χ' on $J^q(A, A')$. Any representative in each equivalence class satisfying the equations

$$D^\mu D_\mu\chi(A) = \eta^{\mu\nu}A_{\mu,\nu}, \quad D^\mu D_\mu\chi'(A') = \eta^{\mu\nu}A'_{\mu,\nu} \quad (5.1)$$

corresponds to the choice of Lorentz gauge being imposed on the potentials. (Note χ, χ' are unique to within the addition of any solution of the source-free wave equation.) Hence we have an embedding of $\tilde{R}(A, A')$ into $R(A, A')$ given by the explicit jet space coordinates $(x^\mu, \tilde{A}_\nu, \tilde{A}'_\nu, F_{\mu\nu}, \tilde{A}_{\mu\nu}, \tilde{A}'_{\mu\nu})$ such that

$$\tilde{A}_\nu = A_\nu - D_\nu\chi(A), \quad \tilde{A}'_\nu = A'_\nu - D_\nu\chi'(A') \quad (5.2)$$

satisfy Lorentz gauge

$$\eta^{\mu\nu} \tilde{A}_{\nu,\mu} = \eta^{\mu\nu} \tilde{A}'_{\nu,\mu} = 0. \quad (5.3)$$

Note here

$$F_{\mu\nu} = \tilde{A}_{[\nu,\mu]} = -*\tilde{A}'_{[\nu,\mu]}, \quad (5.4)$$

and

$$\tilde{A}_{\mu\nu} = \text{trfr}(A_{\mu\nu} - D_\mu D_\nu \chi(A)) = A_{(\nu,\mu)} - D_\mu D_\nu \chi(A), \quad (5.5)$$

$$\tilde{A}'_{\mu\nu} = \text{trfr}(A'_{\mu\nu} - D_\mu D_\nu \chi'(A')) = A'_{(\nu,\mu)} - D_\mu D_\nu \chi'(A'), \quad (5.6)$$

where, we recall, $A_{\mu\nu} = A_{(\nu,\mu)}$, $A'_{\mu\nu} = A'_{(\nu,\mu)}$. Hereafter, $R(\tilde{A}, \tilde{A}') \subset J^1(A, A')$ will denote $\tilde{R}(A, A') \subset J^1(A, A')$ under this embedding of the solution jet spaces.

A. Induced symmetries of Maxwell's equations

Definition V.1 Any local symmetry $\mathbf{X}\tilde{A}_\mu = Q_\mu[\tilde{A}, \tilde{A}']$, $\mathbf{X}\tilde{A}'_\mu = Q'_\mu[\tilde{A}, \tilde{A}']$ of the joint potential system on $R(\tilde{A}, \tilde{A}')$ projects to a symmetry $\mathbf{X}F_{\mu\nu} = D_{[\mu}Q_{\nu]}[A, A']$ of Maxwell's equations on $R(A, A')$ via the transformation (5.2)–(5.6). A projected symmetry is local in the electromagnetic field iff $\mathbf{X}F_{\mu\nu}$ has no essential dependence on the potentials $A_\nu, A'_\nu, A_{\mu\nu}, A'_{\mu\nu}$, so $D_{[\mu}Q_{\nu]}$ is a differential 2-form on $R(F)$; and otherwise a projected symmetry $\mathbf{X}F_{\mu\nu}[A, A']$ is nonlocal.

We now list in Table VIII the projected geometric symmetries obtained from Theorem IV.1. The projections are a straightforward computation using Definition V.1.

The projected symmetries with essential dependence on the potentials are simply the conformal Killing-vector symmetries and their dual symmetries, and the internal rotation/boost symmetries, which comprise the new geometric symmetries found for the joint potential system in Lorentz gauge. Note the Lie derivative term in the conformal Killing-vector symmetries corresponds itself precisely to the local conformal symmetries of Maxwell's equations, and likewise for the Lie derivative term in the dual symmetries. By subtraction of these local symmetries

$$\mathbf{X}_{conf} = (\mathcal{L}_\xi F_{\mu\nu}) \frac{\partial}{\partial F_{\mu\nu}}, \quad \mathbf{X}'_{conf} = (\mathcal{L}_\xi *F_{\mu\nu}) \frac{\partial}{\partial F_{\mu\nu}}, \quad (5.7)$$

Geometric symmetry type	Induced symmetry of $D^\mu F_{\mu\nu} = 0, \quad D^\mu *F_{\mu\nu} = 0$
scaling (4.9) and duality-rotation (4.10)	$\mathbf{X}_{scal} = F_{\mu\nu} \partial / \partial F_{\mu\nu}$ $\mathbf{X}_{dual} = *F_{\mu\nu} \partial / \partial F_{\mu\nu}$
internal rotation/boost (4.11) $\gamma_{\mu\nu}$ given by (4.12)	$\mathbf{X}_\gamma = (\tilde{A}_{[\mu}{}^\sigma \gamma_{\nu]\sigma} + \tilde{A}'_{[\mu}{}^\sigma * \gamma_{\nu]\sigma}) \partial / \partial F_{\mu\nu}$
translation, rotation/boost, dilation (4.13) and dual (4.14) ξ^μ given by (4.15)	$\mathbf{X}_{\xi^H} = (\mathcal{L}_\xi F_{\mu\nu}) \partial / \partial F_{\mu\nu}$ $\mathbf{X}'_{\xi^H} = (\mathcal{L}_\xi *F_{\mu\nu}) \partial / \partial F_{\mu\nu}$
conformal (4.16) and dual (4.17) $\xi^\mu, \zeta^{\mu\nu}, \Omega$ given by (4.18) $\hat{\mathcal{L}}_\xi := \mathcal{L}_\xi + \frac{1}{4}\Omega$	$\mathbf{X}_\xi = (\hat{\mathcal{L}}_\xi F_{\mu\nu} + \tilde{A}_{[\mu}{}^\sigma \zeta_{\nu]\sigma} + \tilde{A}'_{[\mu}{}^\sigma * \zeta_{\nu]\sigma} - \tilde{A}^\sigma \partial_\sigma \zeta_{\mu\nu} + \tilde{A}'^\sigma \partial_\sigma * \zeta_{\mu\nu}) \partial / \partial F_{\mu\nu}$ $\mathbf{X}'_\xi = (\hat{\mathcal{L}}_\xi *F_{\mu\nu} + \tilde{A}'_{[\mu}{}^\sigma \zeta_{\nu]\sigma} - \tilde{A}_{[\mu}{}^\sigma * \zeta_{\nu]\sigma} - \tilde{A}'^\sigma \partial_\sigma \zeta_{\mu\nu} - \tilde{A}^\sigma \partial_\sigma * \zeta_{\mu\nu}) \partial / \partial F_{\mu\nu}$

TABLE VIII: Symmetries of Maxwell's equations induced by geometric symmetries of the joint potential system in Lorentz gauge

we are left with new symmetries \mathbf{Z} of a similar form to the internal rotation/boost symmetries, involving no dependence on derivatives of the electromagnetic field $F_{\mu\nu}$ or its dual, as given by the following transformations:

(i) internal rotation/boost transformation

$$\mathbf{Z}_\gamma = \left(A_{[\mu}{}^\sigma \gamma_{\nu]\sigma} + A'_{[\mu}{}^\sigma * \gamma_{\nu]\sigma} + \gamma_{[\mu}{}^\sigma D_{\nu]} D_\sigma \chi(A) + * \gamma_{[\mu}{}^\sigma D_{\nu]} D_\sigma \chi'(A') \right) \frac{\partial}{\partial F_{\mu\nu}}, \quad (5.8)$$

with $\gamma^{\mu\nu} = \gamma^{[\mu\nu]} = \text{const}$,

(ii) internal conformal transformation

$$\begin{aligned} \mathbf{Z}_\xi = & \left(\frac{1}{4} \Omega F_{\mu\nu} + A_{[\mu}{}^\sigma \zeta_{\nu]\sigma} + A'_{[\mu}{}^\sigma * \zeta_{\nu]\sigma} - A^\sigma \partial_\sigma \zeta_{\mu\nu} + A'^\sigma \partial_\sigma * \zeta_{\mu\nu} \right. \\ & \left. - D_{[\mu} (\zeta_{\nu]}{}^\sigma D_\sigma \chi(A) + * \zeta_{\nu]}{}^\sigma D_\sigma \chi'(A') + \frac{1}{3} \partial^\sigma \zeta_{\nu]\sigma} \chi(A) \right) \frac{\partial}{\partial F_{\mu\nu}}, \end{aligned} \quad (5.9)$$

and dual transformation

$$\begin{aligned} \mathbf{Z}'_\xi = & \left(\frac{1}{4} \Omega * F_{\mu\nu} + A'_{[\mu}{}^\sigma \zeta_{\nu]\sigma} - A_{[\mu}{}^\sigma * \zeta_{\nu]\sigma} - A'^\sigma \partial_\sigma \zeta_{\mu\nu} - A^\sigma \partial_\sigma * \zeta_{\mu\nu} \right. \\ & \left. - D_{[\mu}(\zeta_{\nu]}{}^\sigma D_\sigma \chi'(A') - * \zeta_{\nu]}{}^\sigma D_\sigma \chi(A) + \frac{1}{3} \partial^\sigma \zeta_{\nu]\sigma} \chi'(A')) \right) \frac{\partial}{\partial F_{\mu\nu}}, \end{aligned} \quad (5.10)$$

where $\zeta^{\mu\nu} := -\frac{1}{2} \partial^{[\mu} \xi^{\nu]}$ and $\Omega := \frac{1}{2} \partial_\mu \xi^\mu$, with $\xi^\mu = k_4{}^\sigma x_\sigma x^\mu - \frac{1}{2} k_4{}^\mu x^\sigma x_\sigma$, $k_4{}^\mu = \text{const}$; $\chi(A), \chi'(A')$ are scalar functions satisfying the wave equation (5.1).

Their explicit dependence on the potentials means that these symmetries are nonlocal and nontrivial.

We remark that the transformations (5.9) and (5.10) continue to be admitted as symmetries if the conformal Killing vector ξ^μ is replaced by a homothetic Killing vector (4.15) as seen from Proposition IV.2. Indeed, in the case of a rotation/boost $\xi^\mu = k_2{}^{\mu\nu} x_\nu$, these transformations respectively reduce to the internal rotation/boost symmetries (5.8) with parameters $\gamma^{\mu\nu} = \frac{1}{2} k_2{}^{\mu\nu}, -\frac{1}{4} \epsilon^{\mu\nu}{}_{\sigma\tau} k_2{}^{\sigma\tau}$; in contrast these transformations yield a multiple $\frac{1}{2} k_3$ of the scaling and duality-rotation symmetries in the case of a dilation $\xi^\mu = k_3 x^\mu$, and trivial symmetries in the case of a translation $\xi^\mu = k_1{}^\mu$.

Thus, the nonlocal symmetries we have found for Maxwell's equations arise from new symmetries of the form (5.9) and (5.10) for a general conformal Killing vector i.e. ξ^μ is any generator of a conformal isometry (4.19) of Minkowski space.

There is a deeper unity between the internal rotation/boost symmetries and the internal conformal and dual symmetries. Consider the transformations (5.9) and (5.10) using the sum of a rotation/boost Killing vector and a conformal Killing vector

$$\xi^\mu = k_2{}^{\mu\nu} x_\nu + k_4{}^\sigma x_\sigma x^\mu - \frac{1}{2} k_4{}^\mu x^\sigma x_\sigma. \quad (5.11)$$

We observe that only the (scaled) curl $\zeta^{\mu\nu}$ and divergence Ω of this Killing vector enter these transformations, where Ω is related to $\zeta^{\mu\nu}$ by

$$\zeta^{\mu\nu} = \frac{1}{2} k_2{}^{\mu\nu} - k_4{}^{[\mu} x^{\nu]}, \quad \Omega = 2 k_4{}^\mu x_\mu = \frac{4}{3} x_\nu \partial_\mu \zeta^{\mu\nu}. \quad (5.12)$$

Moreover, $\zeta^{\mu\nu}$ has precisely the form of the dual of a Killing-Yano tensor, namely $Y^{\mu\nu}(x) := * \zeta^{\mu\nu}$ satisfies the Killing-Yano equation [33, 34]

$$\partial^{(\sigma} Y^{\mu)\nu} = 0. \quad (5.13)$$

(More general Killing-Yano tensors of conformal type are parameters for local chiral symmetries of Maxwell's equations [2, 13, 14, 32] and first arose in the study of integrals of the geodesic equations for light rays in the curved Kerr metric; they are also connected with separation of variables of Maxwell's equations in that metric.)

Thus the nonlocal internal symmetries (5.8), (5.9), (5.10) have the following geometric form:

$$\begin{aligned} \mathbf{Z}_Y &= \left(\frac{1}{3} x_\sigma \partial_\tau *Y^{\sigma\tau} F_{\mu\nu} - \tilde{A}_{[\mu}{}^\sigma *Y_{\nu]\sigma} + \tilde{A}'_{[\mu}{}^\sigma Y_{\nu]\sigma} + \tilde{A}_\sigma \partial^\sigma *Y_{\mu\nu} + \tilde{A}'_\sigma \partial^\sigma Y_{\mu\nu} \right) \frac{\partial}{\partial F_{\mu\nu}}, \\ &= \left(\frac{1}{3} x_\sigma \partial_\tau *Y^{\sigma\tau} F_{\mu\nu} - A_{[\mu}{}^\sigma *Y_{\nu]\sigma} + A'_{[\mu}{}^\sigma Y_{\nu]\sigma} + A_\sigma \partial^\sigma *Y_{\mu\nu} + A'_\sigma \partial^\sigma Y_{\mu\nu} \right. \\ &\quad \left. + D_{[\mu}(*Y_{\nu]}{}^\sigma D_\sigma \chi(A) - Y_{\nu]}{}^\sigma D_\sigma \chi'(A') + \frac{1}{3} \partial^\sigma *Y_{\nu]\sigma} \chi(A) \right) \frac{\partial}{\partial F_{\mu\nu}}, \end{aligned} \quad (5.14)$$

$$\begin{aligned} \mathbf{Z}'_Y &= \left(\frac{1}{3} x_\sigma \partial_\tau *Y^{\sigma\tau} *F_{\mu\nu} - \tilde{A}'_{[\mu}{}^\sigma *Y_{\nu]\sigma} - \tilde{A}_{[\mu}{}^\sigma Y_{\nu]\sigma} + \tilde{A}'_\sigma \partial^\sigma *Y_{\mu\nu} - \tilde{A}_\sigma \partial^\sigma Y_{\mu\nu} \right) \frac{\partial}{\partial F_{\mu\nu}}, \\ &= \left(\frac{1}{3} x_\sigma \partial_\tau *Y^{\sigma\tau} *F_{\mu\nu} - A'_{[\mu}{}^\sigma *Y_{\nu]\sigma} - A_{[\mu}{}^\sigma Y_{\nu]\sigma} + A'_\sigma \partial^\sigma *Y_{\mu\nu} - A_\sigma \partial^\sigma Y_{\mu\nu} \right. \\ &\quad \left. + D_{[\mu}(*Y_{\nu]}{}^\sigma D_\sigma \chi'(A') + Y_{\nu]}{}^\sigma D_\sigma \chi(A) + \frac{1}{3} \partial^\sigma *Y_{\nu]\sigma} \chi'(A') \right) \frac{\partial}{\partial F_{\mu\nu}}, \end{aligned} \quad (5.15)$$

where

$$Y^{\mu\nu} = \bar{k}_1^{\mu\nu} + \epsilon^{\mu\nu}{}_{\sigma\tau} \bar{k}_2{}^\sigma x^\tau, \quad \bar{k}_1^{\mu\nu} = \bar{k}_1^{[\mu\nu]}, \bar{k}_2{}^\sigma = \text{const} \quad (5.16)$$

is a Killing-Yano tensor. Note that only the set of constant Killing-Yano tensors is preserved under duality $Y^{\mu\nu} \rightarrow *Y^{\mu\nu}$.

Theorem V.2 *Maxwell's equations admits the (nontrivial) nonlocal symmetries (5.14) and (5.15) depending on an arbitrary Killing-Yano tensor (5.16). Under the duality transformation (2.11) on the potentials, these symmetries are interchanged, and in the case of a constant Killing-Yano tensor they are related through directly replacing this tensor with its dual. Thus the symmetries comprise a 14-dimensional vector space.*

The vector space structure of these nonlocal symmetries has a basis consisting of six of internal rotation/boost type (5.8), four of internal conformal type (5.9) and four of dual type (5.10). However, their commutator structure is not closed. Note the internal rotation/boost symmetries themselves comprise a $\mathfrak{so}(3, 1)$ Lie algebra, which follows from projection of the commutator structure stated in Theorem IV.3 for the corresponding local symmetries on the solution jet space $\tilde{R}(A, A')$ of the joint potential system in Lorentz gauge. In contrast

the internal conformal symmetries and their duals do not arise from projection of any local symmetries of this potential system, and as a consequence, their commutators do not have the form of local transformations on $\tilde{R}(A, A')$, i.e. they necessarily involve a nonlocal dependence on the potentials and hence there is no natural Lie algebra structure for them. The same conclusion holds for the commutators of the internal rotation/boost symmetries with the internal conformal symmetries or their duals. These commutators are still themselves nonlocal symmetries of the potential system and hence of Maxwell's equations. More generally, there is an enveloping (nonlocal) symmetry algebra generated by the span of the symmetries (5.8), (5.9), (5.10), and their repeated commutators.

If instead we consider the respective superpositions $\mathbf{Z}_\xi + \mathbf{X}_{conf}$, $\mathbf{Z}'_\xi + \mathbf{X}'_{conf}$, of the nonlocal symmetries (5.9), (5.10) of Maxwell's equations and the corresponding local symmetries (5.7) given by rotation/boost Killing vectors and conformal Killing vectors, their collective enveloping algebra collapses to the Lie algebra of rotations/boosts and complexified inversions, namely the natural semidirect product of $\mathfrak{so}(3, 1)$ with $\mathfrak{u}(1)^4 \otimes \mathbb{C}$, as follows from Theorem IV.3.

B. Induced conservation laws

Definition V.3 *Any local conserved current $\Psi^\mu[\tilde{A}, \tilde{A}']$ on $R(\tilde{A}, \tilde{A}')$ directly projects to a conserved current $\Psi^\mu[A, A']$ of Maxwell's equations on $R(A, A')$ via the transformation (5.2)–(5.6). A projected current is local in the electromagnetic field iff, up to addition of a curl, it has no essential dependence on the potentials $A_\nu, A'_\nu, A_{\mu\nu}, A'_{\mu\nu}$, so that $\Psi^\mu[A, A'] + D_\nu \Theta^{\mu\nu}[A, A']$ for some antisymmetric tensor function $\Theta^{\mu\nu}[A, A']$ is a vector function on $R(F)$; and otherwise a projected current $\Psi^\mu[A, A']$ is nonlocal.*

For the geometric conserved currents listed in Table VII for the joint potential system in Lorentz gauge, the Lie derivative terms in the general conformal Killing-vector currents simply project to the corresponding stress-energy currents (4.60), while the Lie derivative terms in the dual currents project to a curl. The remaining internal terms in those currents as well as the internal rotation/boost currents and duality-rotation current separately project to new conserved currents Φ^μ of Maxwell's equations:

(i) duality-rotation conservation law

$$\Phi_{dual}^\mu = \tilde{A}'_\nu F^{\mu\nu} - \tilde{A}_\nu * F^{\mu\nu} = A'_\nu F^{\mu\nu} - A_\nu * F^{\mu\nu} \text{ mod curls,} \quad (5.17)$$

(ii) internal rotation/boost conservation law

$$\begin{aligned} \Phi_\gamma^\mu &= \gamma_\nu{}^\sigma (\tilde{A}_\sigma F^{\mu\nu} + \tilde{A}'_\sigma * F^{\mu\nu}) + * \gamma_\nu{}^\sigma (\tilde{A}'_\sigma F^{\mu\nu} - \tilde{A}_\sigma * F^{\mu\nu}) \\ &= -\frac{1}{2} \gamma^{\nu\sigma} (A^\mu F_{\nu\sigma} + A'^\mu * F_{\nu\sigma}) + 2 \gamma^{\nu[\sigma} (A_\sigma F^{\mu]}{}_\nu + A'_\sigma * F^{\mu]}{}_\nu) \\ &\quad + \Upsilon^\mu(\chi(A), \gamma^{\nu\sigma} F_{\nu\sigma}) + \Upsilon^\mu(\chi'(A'), \gamma^{\nu\sigma} * F_{\nu\sigma}) \\ &\quad \text{mod curls,} \end{aligned} \quad (5.18)$$

with $\gamma^{\mu\nu} = \gamma^{[\mu\nu]} = \text{const}$,

(iii) internal conformal conservation law

$$\begin{aligned} \Phi_\xi^\mu &= \zeta_\nu{}^\sigma (\tilde{A}_\sigma F^{\mu\nu} + \tilde{A}'_\sigma * F^{\mu\nu}) + * \zeta_\nu{}^\sigma (\tilde{A}'_\sigma F^{\mu\nu} - \tilde{A}_\sigma * F^{\mu\nu}) + \frac{1}{4} \Omega (\tilde{A}_\nu F^{\mu\nu} + \tilde{A}'_\nu * F^{\mu\nu}) \\ &= -\frac{1}{2} \zeta^{\nu\sigma} (A^\mu F_{\nu\sigma} + A'^\mu * F_{\nu\sigma}) + 2 \zeta^{\nu[\sigma} (A_\sigma F^{\mu]}{}_\nu + A'_\sigma * F^{\mu]}{}_\nu) - \frac{1}{8} \epsilon^{\mu\sigma\alpha\beta} \partial_\sigma \Omega A_\alpha A'_\beta \\ &\quad + \Upsilon^\mu(\chi(A), \zeta^{\nu\sigma} F_{\nu\sigma}) + \Upsilon^\mu(\chi'(A'), \zeta^{\nu\sigma} * F_{\nu\sigma}) \\ &\quad \text{mod curls,} \end{aligned} \quad (5.19)$$

and dual conservation law

$$\begin{aligned} \Phi'_\xi{}^\mu &= \zeta_\nu{}^\sigma (\tilde{A}'_\sigma F^{\mu\nu} - \tilde{A}_\sigma * F^{\mu\nu}) - * \zeta_\nu{}^\sigma (\tilde{A}_\sigma F^{\mu\nu} + \tilde{A}'_\sigma * F^{\mu\nu}) + \frac{1}{4} \Omega (\tilde{A}'_\nu F^{\mu\nu} - \tilde{A}_\nu * F^{\mu\nu}) \\ &= -\frac{1}{2} \zeta^{\nu\sigma} (A'^\mu F_{\nu\sigma} - A^\mu * F_{\nu\sigma}) + 2 \zeta^{\nu[\sigma} (A'_\sigma F^{\mu]}{}_\nu - A_\sigma * F^{\mu]}{}_\nu) \\ &\quad + \frac{1}{4} \Omega (A'_\nu F^{\mu\nu} - A_\nu * F^{\mu\nu}) - \Upsilon^\mu(\chi(A), \zeta^{\nu\sigma} * F_{\nu\sigma}) + \Upsilon^\mu(\chi'(A'), \zeta^{\nu\sigma} F_{\nu\sigma}) \\ &\quad \text{mod curls,} \end{aligned} \quad (5.20)$$

where $\zeta^{\mu\nu} := -\frac{1}{2} \partial^{[\mu} \xi^{\nu]}$ and $\Omega := \frac{1}{2} \partial_\mu \xi^\mu$, with $\xi^\mu = k_4^\sigma x_\sigma x^\mu - \frac{1}{2} k_4^\mu x^\sigma x_\sigma$, $k_4^\mu = \text{const}$; here $\chi(A), \chi'(A')$ are scalar functions satisfying the wave equation (5.1), and $\Upsilon^\mu(f, g) := \frac{1}{2} (g D^\mu f - f D^\mu g)$ is a skew-bilinear vector function depending on any scalar expressions f, g .

In writing downs these currents we have simplified some terms through integration by parts and multiplied by an overall factor of 2. We remark that $\Upsilon^\mu(f, g)$ has the form of a conserved current formula for the ordinary scalar wave equation [1], taking f, g to be a pair of symmetries. These terms are not separately conserved in the currents here.

Note that if the conformal Killing vector ξ^μ in the currents (5.19) is replaced by a homothetic Killing vector (4.15), we obtain the internal rotation/boost currents (5.18) with $\gamma^{\mu\nu} = \frac{1}{2}k_2^{\mu\nu}$ in the case of a rotation/boost $\xi^\mu = k_2^{\mu\nu}x_\nu$, and trivial currents in the case of a translation $\xi^\mu = k_1^\mu$ or a dilation $\xi^\mu = x^\mu$. Similarly, from the dual currents (5.20) we obtain the internal rotation/boost current (5.18) with $\gamma^{\mu\nu} = -\frac{1}{4}\epsilon^{\mu\nu}{}_{\alpha\beta}k_2^{\alpha\beta}$ in the case of a rotation/boost $\xi^\mu = k_2^{\mu\nu}x_\nu$, the duality-rotation current (5.17) in the case of a dilation $\xi^\mu = x^\mu$, and a trivial current in the case of translation $\xi^\mu = k_1^\mu$.

Thus, the new currents we have found for Maxwell's equations come from new conservation laws of the form (5.19) and (5.20) that exist for a general conformal Killing vector, i.e. ξ^μ is any generator of a conformal isometry (4.19) of Minkowski space.

We now prove these conservation laws (5.17)–(5.20) are nontrivial and nonlocal.

Firstly, it is useful to define the weight of a quadratic current Ψ^μ on $J^q(A, A')$ to be the maximum of the weights of all monomial terms in Ψ^μ , given by counting the total number of derivatives that appear on the potentials: i.e. A_μ, A'_μ have weight 0, $F_{\mu\nu}$ has weight 1, while $\chi(A), \chi'(A')$ are counted as weight -1 through equation (5.1). The lowest weight nontrivial *local* conserved currents of Maxwell's equations as shown in Ref.[3] are the stress-energy currents (3.90), which have weight 2 on $J(F) \subset J^1(A, A')$. In comparison, the currents (5.17)–(5.20) each have weight 1 and cannot be equivalent consequently to any nontrivial local current of Maxwell's equations. This establishes, moreover, that these currents (5.17)–(5.20) are nonlocal. Thus it remains to show only that they are nontrivial when restricted to the solution jet space $R(\tilde{A}, \tilde{A}')$.

Lemma V.4 *Suppose*

$$\Psi^\mu = (k_1^\mu{}_{\alpha\beta\sigma}\tilde{A}^\sigma + k_2^\mu{}_{\alpha\beta\sigma}\tilde{A}'^\sigma)F^{\alpha\beta} \quad (5.21)$$

is a conserved current on $R(\tilde{A}, \tilde{A}')$. Then this current is trivial iff

$$k_2^{\mu\alpha\beta\sigma} = \frac{1}{2}\epsilon_{\nu\tau}{}^{\alpha\beta}k_1^{\mu\nu\tau\sigma} = k\epsilon^{\mu\alpha\beta\sigma}, \quad k = \text{const} \quad (5.22)$$

Proof: Any trivial current on $R(\tilde{A}, \tilde{A}')$ is characterized on $J^1(A, A')$ by having the form

$$\Psi^\mu = D_\nu\Theta^{\mu\nu} + b^\mu{}_{\alpha\beta}(F'^{\alpha\beta} - *F^{\alpha\beta}), \quad F'^{\alpha\beta} := \tilde{A}'^{[\beta,\alpha]}, \quad F^{\alpha\beta} := \tilde{A}^{[\beta,\alpha]}, \quad (5.23)$$

where

$$\Theta^{\mu\nu} = \Theta_1^{\mu\nu}{}_{\alpha\beta}(x)\tilde{A}^\alpha\tilde{A}^\beta + \Theta_2^{\mu\nu}{}_{\alpha\beta}(x)\tilde{A}'^\alpha\tilde{A}'^\beta + \Theta'^{\mu\nu}{}_{\alpha\beta}(x)\tilde{A}'^\alpha\tilde{A}^\beta, \quad (5.24)$$

$$b^\mu{}_{\alpha\beta} = b_1^\mu{}_{\alpha\beta\sigma}(x)\tilde{A}^\sigma + b_2^\mu{}_{\alpha\beta\sigma}(x)\tilde{A}'^\sigma, \quad (5.25)$$

with the coefficients subject to the index symmetries $\Theta_i^{\mu\nu}{}_{\alpha\beta} = \Theta_i^{[\mu\nu]}{}_{(\alpha\beta)}$, $\Theta'^{\mu\nu}{}_{\alpha\beta} = \Theta'^{[\mu\nu]}{}_{\alpha\beta}$, $b_i^\mu{}_{\alpha\beta\sigma} = b_i^\mu{}_{[\alpha\beta]\sigma}$. To proceed we expand (5.23) and substitute the decompositions $\tilde{A}^{\beta,\alpha} = F^{\alpha\beta} + \text{trfr}\tilde{A}^{(\beta,\alpha)}$ and $\tilde{A}'^{\beta,\alpha} = F'^{\alpha\beta} + \text{trfr}\tilde{A}'^{(\beta,\alpha)}$. The coefficients of all terms other than $F^{\alpha\beta}\tilde{A}^\sigma$ and $F'^{\alpha\beta}\tilde{A}'^\sigma$ cannot match (5.21) and must therefore vanish, which leads to the conditions

$$\Theta_1^\mu{}_{(\nu\alpha)\beta} = \Theta_2^\mu{}_{(\nu\alpha)\beta} = \Theta'^\mu{}_{(\nu|\beta|\alpha)} = \Theta'^\mu{}_{(\nu\alpha)\beta} = 0, \quad (5.26)$$

$$2\Theta_2^\mu{}_{[\alpha\beta]\sigma} + b_2^\mu{}_{\alpha\beta\sigma} = 0, \quad (5.27)$$

$$\Theta'^\mu{}_{[\alpha\beta]\sigma} + b_1^\mu{}_{\alpha\beta\sigma} = 0. \quad (5.28)$$

The index symmetries imposed by (5.26) immediately show that $\Theta_1^\mu{}_{\nu\alpha\beta}$ and $\Theta_2^\mu{}_{\nu\alpha\beta}$ vanish while $\Theta'^\mu{}_{\nu\alpha\beta}$ must be totally antisymmetric and hence

$$\Theta'^\mu{}_{\nu\alpha\beta} = k(x)\epsilon^\mu{}_{\nu\alpha\beta} \quad (5.29)$$

for some $k(x)$. It then follows from (5.27) that $b_2^\mu{}_{\alpha\beta\sigma}$ also vanishes. Finally, by now equating (5.21) to (5.23) and collecting like terms, we obtain

$$k_1^\mu{}_{\alpha\beta\sigma} = -\frac{1}{2}\epsilon_{\alpha\beta}{}^{\rho\tau}b_1^\mu{}_{\rho\tau\sigma}, \quad k_2^\mu{}_{\rho\tau\sigma} = -\Theta'^\mu{}_{\rho\tau\sigma}, \quad \partial^\rho k_2^\mu{}_{\rho\tau\sigma} = 0. \quad (5.30)$$

Then the relations (5.28) and (5.30) establish the conditions (5.22) stated in the Lemma. \square

To apply this Lemma, we observe the new conserved currents (5.17)–(5.20) written in terms of $\tilde{A}_\nu, \tilde{A}'_\nu, F_{\mu\nu}$ have the form (5.21) where

$$k_{1\mu\alpha\beta\sigma} = \eta_{\mu[\alpha}a_{\beta]\sigma} - \frac{1}{2}\epsilon_{\mu\alpha\beta}{}^\nu a'_{\nu\sigma}, \quad k_{2\mu\alpha\beta\sigma} = \eta_{\mu[\alpha}a'_{\beta]\sigma} + \frac{1}{2}\epsilon_{\mu\alpha\beta}{}^\nu a_{\nu\sigma} \quad (5.31)$$

as given by Table IX. The algebraic conditions (5.22) are readily found to fail for each of these currents. Therefore our main result is now proven.

Theorem V.5 *Maxwell's equations admits the (nontrivial) nonlocal conservation laws (5.17)–(5.20). These conservation laws span a 15-dimensional vector space whose basis consists of one of duality-rotation type, six of internal rotation/boost type, four of internal conformal type and four of dual type.*

Conserved current	$a_{\mu\nu}$	$a'_{\mu\nu}$
Duality-rotation	0	$\eta_{\mu\nu}$
Internal rotation/boost	$\gamma_{\mu\nu}$	$*\gamma_{\mu\nu}$
Internal conformal	$\zeta_{\mu\nu} + \frac{1}{4}\Omega\eta_{\mu\nu}$	$*\zeta_{\mu\nu}$
Internal dual conformal	$-\zeta_{\mu\nu}$	$\zeta_{\mu\nu} + \frac{1}{4}\Omega\eta_{\mu\nu}$

TABLE IX: Form of nonlocal currents of Maxwell's equations

We remark that these nonlocal conservation laws also arise directly from the nonlocal symmetries (5.8)–(5.10) through use of the formula (3.87) in Theorem III.16 that generates conserved currents of Maxwell's equations from symmetries of Maxwell's equations. As a consequence, firstly, all the currents (5.17)–(5.20) are invariant under the duality transformation (2.11) on the potentials (with the simultaneous induced transformation (2.4) on the electromagnetic field). More significantly, the internal conformal currents and dual currents can be unified with the internal rotation/boost currents by the introduction of a Killing-Yano tensor, corresponding to the analogous form of the nonlocal symmetries (5.14)–(5.15) stated in Theorem V.2. As we recall, the Killing-Yano tensor is identified with the dual of the curl of a Killing vector of the form (5.11) given by the sum of a rotation/boost Killing vector and a conformal Killing vector.

Corollary V.6 *The nonlocal conservation laws (5.18), (5.19), (5.20) admitted by Maxwell's equations have the unified form*

$$\begin{aligned}
\Phi_Y^\mu &= Y_\nu{}^\sigma (\tilde{A}'_\sigma F^{\mu\nu} - \tilde{A}_\sigma *F^{\mu\nu}) - *Y_\nu{}^\sigma (\tilde{A}_\sigma F^{\mu\nu} + \tilde{A}'_\sigma *F^{\mu\nu}) \\
&\quad + \frac{1}{3}x^\sigma \partial^\tau *Y_{\sigma\tau} (\tilde{A}_\nu F^{\mu\nu} + \tilde{A}'_\nu *F^{\mu\nu}) \\
&= \frac{1}{2}Y^{\nu\sigma} (A^\mu *F_{\nu\sigma} - A'^\mu F_{\nu\sigma}) - 2Y^{\nu[\sigma} (A_\sigma *F^{\mu]}{}_\nu - A'_\sigma F^{\mu]}{}_\nu) + \frac{1}{6}\partial^\tau *Y_{\tau\nu} \epsilon^{\mu\nu\alpha\beta} A_\alpha A'_\beta \\
&\quad - \Upsilon^\mu(\chi(A), Y^{\nu\sigma} *F_{\nu\sigma}) + \Upsilon^\mu(\chi'(A'), Y^{\nu\sigma} F_{\nu\sigma}) \\
&\quad \text{mod curls,}
\end{aligned} \tag{5.32}$$

and

$$\begin{aligned}
\Phi_Y'^\mu &= -Y_\nu{}^\sigma (\tilde{A}_\sigma F^{\mu\nu} + \tilde{A}'_\sigma * F^{\mu\nu}) - *Y_\nu{}^\sigma (\tilde{A}'_\sigma F^{\mu\nu} - \tilde{A}_\sigma * F^{\mu\nu}) \\
&\quad + \frac{1}{3} x^\sigma \partial^\tau *Y_{\sigma\tau} (\tilde{A}'_\nu F^{\mu\nu} - \tilde{A}_\nu * F^{\mu\nu}) \\
&= \frac{1}{2} Y^{\nu\sigma} (A^\mu F_{\nu\sigma} + A'^\mu * F_{\nu\sigma}) - 2Y^{\nu[\sigma} (A_\sigma F^{\mu]}{}_\nu + A'_\sigma * F^{\mu]}{}_\nu) \\
&\quad - \frac{1}{3} x^\sigma \partial^\tau *Y_{\sigma\tau} (A_\nu * F^{\mu\nu} - A'_\nu F^{\mu\nu}) \\
&\quad - \Upsilon^\mu(\chi(A), Y^{\nu\sigma} F_{\nu\sigma}) - \Upsilon^\mu(\chi'(A'), Y^{\nu\sigma} * F_{\nu\sigma}) \\
&\quad \text{mod curls,}
\end{aligned} \tag{5.33}$$

depending on a Killing-Yano tensor (5.16). Under the duality transformation (2.11), (2.4) on the potentials and electromagnetic field, the currents (5.32) and (5.33) are invariant. In the case when the Killing-Yano tensor is constant they are interchanged under replacing $Y^{\mu\nu}$ with its dual $*Y^{\mu\nu}$. Thus these currents span a 14-dimensional vector space which is duality-invariant.

VI. CONCLUDING REMARKS

In conclusion we mention a few applications of our main results.

The new nonlocal infinitesimal symmetries we have obtained for Maxwell's equations in Minkowski space are of point-type [25] to within a duality transformation, when expressed in terms of the joint electric and magnetic potentials for the electromagnetic field. (Consequently, they can be realized as genuine point transformations on the complexified jet space of the potential system.) Thus these symmetries can be used to derive corresponding new group-invariant solutions of Maxwell's equations and to generalize physically interesting solutions (e.g. plane waves and monopoles) under the action of the finite symmetry group of transformations.

The associated new nonlocal conserved currents we have derived from these symmetries give rise to constants of motion for the electromagnetic field. Explicit expressions for them given by gauge-invariant integrals of the field can be obtained by extending the methods used in Ref.[17] to 3+1 dimensions. In particular, we expect the resulting constants of motion to be functionally independent of energy, momentum, angular/boost momentum, and conformal quantities as well as chiral quantities that arise as constants of motion from the local conserved currents of Maxwell's equations.

For future work, it is planned to extend our results to classify all nonlocal symmetries and nonlocal conserved currents of Maxwell's equations produced via a complete classification of symmetries and conserved currents of local form in the potentials and their derivatives to any finite order, admitted by the joint potential system in Lorentz gauge. A further extension of obvious interest would be to derive nonlocal symmetries and nonlocal conserved currents of the electromagnetic field on curved background spacetimes, for instance the Schwarzschild and Kerr black hole spacetimes. This analysis is tractable using the spinor techniques of Ref.[3, 35].

Finally, our methods in this paper readily apply to other linear physical field equations, notably the linearized gravity wave equation in Minkowski space and its spin s generalization. A classification of all local symmetries and local conserved currents for the linear spin $s > 0$ field equations in Minkowski space has been carried out in Ref.[2, 32, 35]. A systematic investigation of spin s potential systems would be of significant interest.

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- [1] S. C. Anco and G. Bluman, J. Math. Phys. **37**, 2361 (1996).
 - [2] S. C. Anco and J. Pohjanpelto, CRM Proceedings and Lecture Notes **34**, 1 (2004).
 - [3] S. C. Anco and J. Pohjanpelto, Acta Applic. Math. **69**, 285 (2001).
 - [4] S. C. Anco, J. Phys. A: Math. Gen. **36**, 8623 (2003).
 - [5] E. Bessel-Hagen, Math. Ann. **84**, 258 (1912).
 - [6] D. Lipkin, J. Math. Phys. **5**, 696 (1964).
 - [7] T. Kibble, J. Math. Phys. **6**, 1022 (1965).
 - [8] D. Fairlie, Nuovo Cimento **37**, 897 (1965).
 - [9] T. Morgan, J. Math. Phys. **5**, 1659 (1964).
 - [10] H. Bateman, Proc. London Math. Soc. **8**, 223 (1909).
 - [11] E. Cunningham, Proc. London Math. Soc. **8**, 77 (1909).
 - [12] N. K. Ibragimov, Sov. Phys. Dokl. **13**, 18 (1968).

- [13] W. Fushchich and A. Nikitin, *Sov. J. Part. Nucl.* **14**, 1 (1983).
- [14] W. Fushchich and A. Nikitin, *Symmetries of Maxwell's Equations* (D. Reidel, Boston, 1987).
- [15] W. Fushchich and A. Nikitin, *J. Phys. A: Math. Gen.* **25**, L231 (1992).
- [16] J. Pohjanpelto, in *Advanced Electromagnetism: Foundations, Theory and Applications*, edited by T. Barrett and D. Grimes (World Scientific, Singapore, 1995), pp. 560–589.
- [17] S. C. Anco and G. Bluman, *J. Math. Phys.* **38**, 3508 (1997).
- [18] G. W. Bluman, S. Kumei, and G. J. Reid, *J. Math. Phys.* **29**, 806 (1988).
- [19] G. Bluman, in *Applications of Analytic and Geometric Methods to Nonlinear Differential Equations*, edited by P. Clarkson (Kluwer, 1993), pp. 363–373.
- [20] G. Bluman and P. Doran-Wu, *Acta Appl. Math.* **41**, 21 (1995).
- [21] I. Krasilshchik and A. Vinogradov, *Sov. Math. Dokl.* **29**, 337 (1984).
- [22] I. Krasilshchik and A. Vinogradov, *Acta Appl. Math.* **2**, 79 (1984).
- [23] I. Krasilshchik and A. Vinogradov, *Acta Appl. Math.* **15**, 161 (1989).
- [24] P. J. Olver, *Applications of Lie Groups to Differential Equations*, vol. 107 of *Graduate Texts in Mathematics* (Springer Verlag, 2000), 2nd ed.
- [25] G. W. Bluman and S. C. Anco, *Symmetry and Integration Methods for Differential Equations*, vol. 154 of *Applied Mathematical Sciences* (Springer Verlag, 2002).
- [26] I. M. Anderson and C. G. Torre, *Commun. Math. Phys.* **176**, 479 (1996).
- [27] S. C. Anco and G. Bluman, *Phys. Rev. Lett.* **78**, 2869 (1997).
- [28] S. C. Anco and G. Bluman, *Eur. J. Appl. Math.* **13**, 567 (2002).
- [29] I. Anderson, *Contemp. Math.* **132**, 51 (1992).
- [30] C. G. Torre (1997), lectures given at the Second Mexican School on Gravitation and Mathematical Physics held in Tlaxcala, Mexico from December 1–7, 1996.
- [31] D. The, Master's thesis, University of British Columbia (2003).
- [32] S. C. Anco and J. Pohjanpelto (2002), in Preparation.
- [33] W. Dietz and R. Rüdiger, *Gen. Rel. Grav.* **12**, 545 (1980).
- [34] W. Dietz and R. Rüdiger, *Proc. Roy. Soc. A* **375**, 361 (1981).
- [35] S. C. Anco and J. Pohjanpelto, *Proc. Roy. Soc. Lond. A* **459**, 1215 (2003).